AREAS BETWEEN CURVES

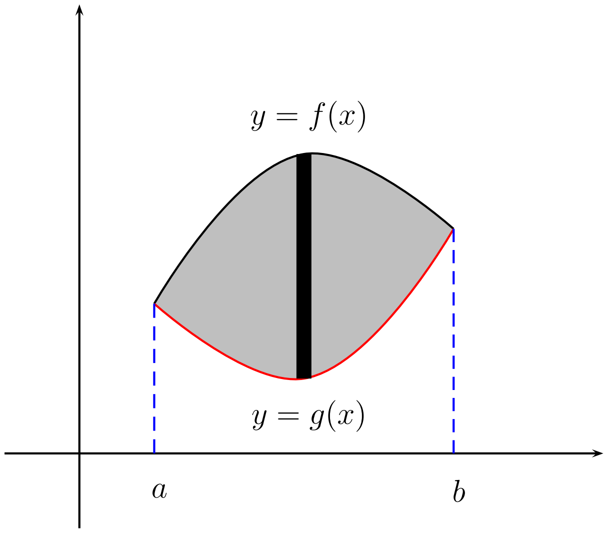

FIGURE 1. area between $y = f(x)$ and $y = g(x)$

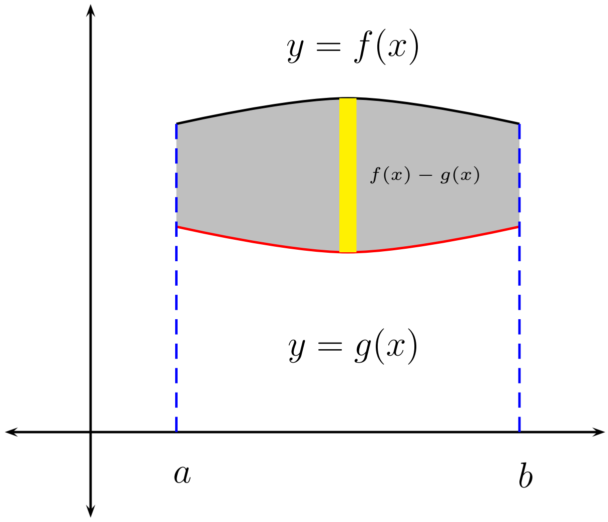

FIGURE 2. area between $y = f(x)$ and $y = g(x)$

Definition 0.1. [Area between $y = f(x)$ and $y = g(x)$:] If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the ***Area*** of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by $x = b$ [see figures 1 and 2] is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

We usually use the notation

$$A = \int_a^b [y_{\text{Top}} - y_{\text{Bottom}}]\, dx.$$

Example 0.1. Find the area of the region bounded above by $y = x^2 + 3$, bounded below by $y = x$, and bounded on the sides by the lines $x = -1$ and $x = 1$.

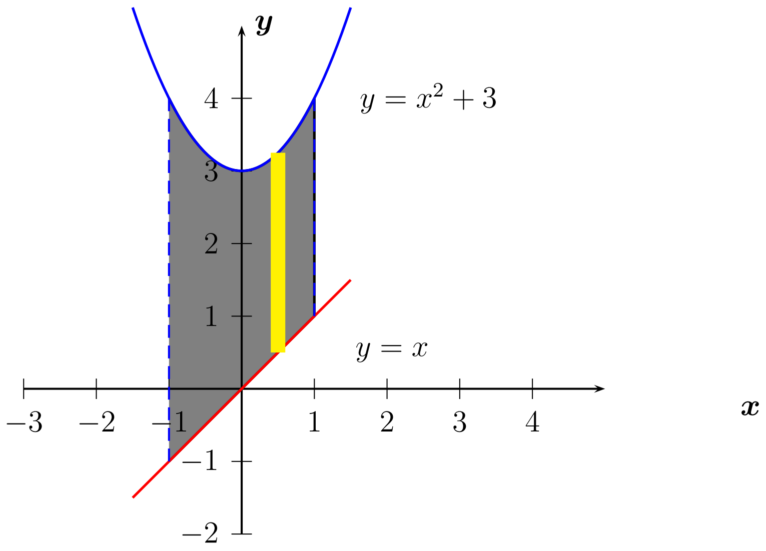

FIGURE 3

Solution:

Here the top curve is $y = x^2 + 3$ and the bottom curve is $y = x$. Also we have $a = -1$ and $b = 1$. Hence

$$\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\, dx \\
&= \int_{-1}^1 [x^2 + 3 - x]\, dx \\
&= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right]_{-1}^1 \\
&= \left[\frac{1}{3} - \frac{1}{2} + 3\right] - \left[\frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2 + 3(-1)\right] \\
&= \left[\frac{1}{3} - \frac{1}{2} + 3 + \frac{1}{3} + \frac{1}{2} + 3\right] \\
&= \left[6 + \frac{2}{3}\right] \\
&= \frac{20}{3}.
\end{aligned}$$

Example 0.2. Find the area of the region bounded by $y = 6 - x^2$, and by $y = x$.

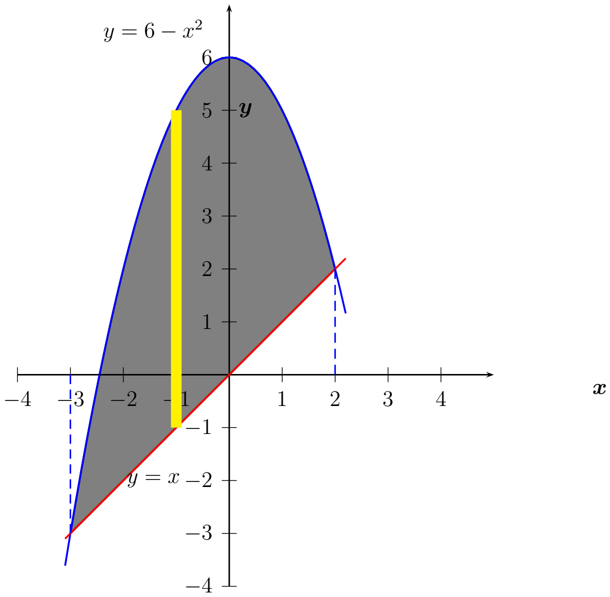

FIGURE 4

Solution:

Here the top curve is $y = 6 - x^2$ and the bottom curve is $y = x$. The limits a and b will be the $x-$coordinates of the intersections of the two curves. To find the limits we set

$$6 - x^2 = x$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3, 2$$

Thus we have $a = -3$ and $b = 2$. Hence

$$\begin{aligned}
A &= \int_a^b [f(x) - g(x)]\ dx \\
&= \int_{-3}^{2} [6 - x^2 - x]\ dx \\
&= \left[6x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-3}^{2} \\
&= \left[6(2) - \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2\right] - \left[6(-3) - \frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2\right] \\
&= \left[12 - \frac{8}{3} - \frac{4}{2} + 18 - \frac{27}{3} + \frac{9}{2}\right] \\
&= \left[19 - \frac{8}{3} + \frac{9}{2}\right] \\
&= \frac{125}{6}.
\end{aligned}$$

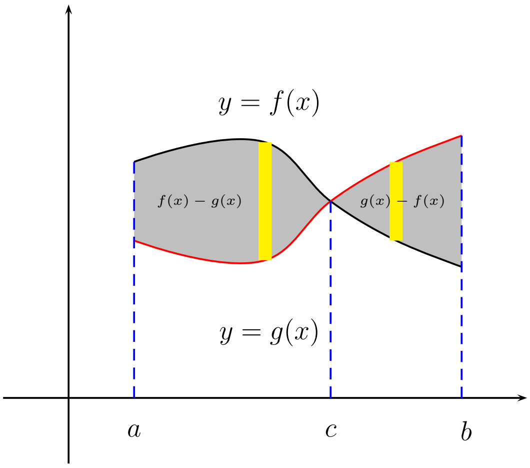

FIGURE 5

Definition 0.2. [Area between $y = f(x)$ and $y = g(x)$:] If f and g are continuous functions on the interval $[a, b]$. Let $a \le c \le b$. Suppose that $f(x) \ge g(x)$ for all x in $[a, c]$, and $g(x) \ge f(x)$ for all x in $[c, b]$. Then the ***Area*** of the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$, and $x = b$ [see figure5] is

$$A = \int_a^c [f(x) - g(x)]\, dx + \int_c^b [g(x) - f(x)]\, dx.$$

Example 0.3. Find the area of the region bounded by the curves $y = \sin x,\ y = \cos x,\ x = 0,$ and $x = \frac{\pi}{2}$.

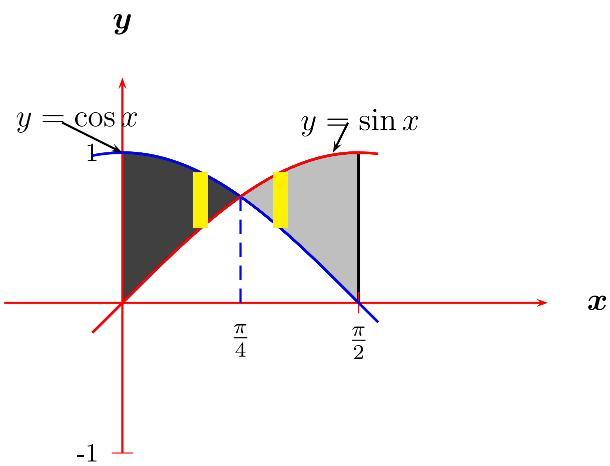

FIGURE 6

Solution:

The two curves intersect when $\sin x = \cos x$, that is, when $x = \frac{\pi}{4}$. Now, since $\cos x \geq \sin x$ when $0 \leq x \leq \frac{\pi}{4}$ and $\sin x \geq \cos x$ when $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$. Then the area is

$$
\begin{aligned}
A =& \int_0^{\frac{\pi}{4}} [\cos x - \sin x] \; dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x] \; dx \\
=& [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\sin x - \cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
=& \left[(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (1 - 0) \right] + \left[(-0 - 1) - (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}) \right] \\
=& \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] \\
=& \frac{4}{\sqrt{2}} - 2 \\
=& 2\sqrt{2} - 2.
\end{aligned}
$$

Example 0.4. Find the area of the region enclosed by $y^2 = x$ and $x - 2y = 3$.

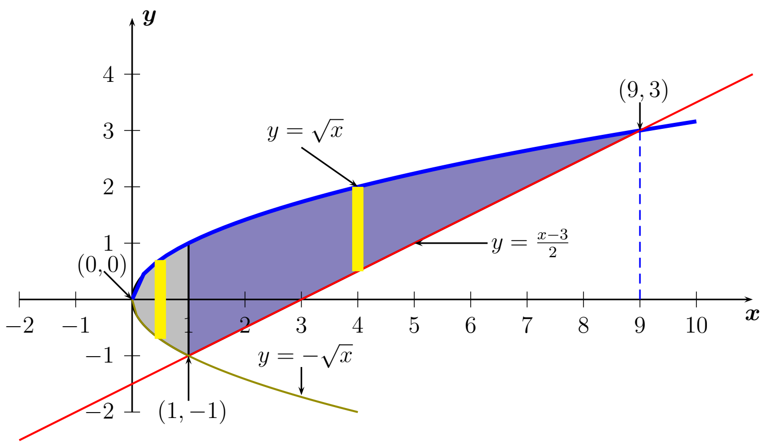

FIGURE 7

Solution: We start with solving the equations $y^2 = x$ and $x - 2y = 3$.

$$y^2 - 2y = 3$$

$$y^2 - 2y - 3 = 0$$

$$(y-3)(y+1) = 0$$

$$y = 3, -1$$

$$\text{Thus } x = 9, 1.$$

Hence the points of intersection are $(1, -1)$ and $(9, 3)$. Knowing the points of intersection does not give us the right limits for the integral. Observe that x moves from 0 to 9. When $0 \leq x \leq 1$, the top curve is $y = \sqrt{x}$ and the bottom curve is $y = -\sqrt{x}$, and when $1 \leq x \leq 9$, the top curve is $y = \sqrt{x}$ and the bottom curve is $y = \frac{x-3}{2}$. Hence

$$\begin{aligned}
A &= \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] \, dx + \int_1^9 \left[\sqrt{x} - \left(\frac{x-3}{2}\right)\right] \, dx \\
&= \int_0^1 \left[2x^{\frac{1}{2}}\right] \, dx + \int_1^9 \left[x^{\frac{1}{2}} - \frac{1}{2}x + \frac{3}{2}\right] \, dx \\
&= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2 + \frac{3}{2}x\right]_1^9 \\
&= \left[\frac{4}{3}\right] + \left[\left(\frac{54}{3} - \frac{81}{4} + \frac{27}{2}\right) - \left(\frac{2}{3} - \frac{1}{4} + \frac{3}{2}\right)\right] \\
&= \left[\frac{4}{3} + \frac{54}{3} - \frac{81}{4} + \frac{27}{2} - \frac{2}{3} + \frac{1}{4} - \frac{3}{2}\right] \\
&= \left[\frac{4}{3} + 18 - 20 + 12 - \frac{2}{3}\right] \\
&= \left[10 + \frac{2}{3}\right] \\
&= \frac{32}{3}.
\end{aligned}$$

There is an easier way for solving Example 0.4. Instead of regarding y as a function of x, we will regard x as a function of y.

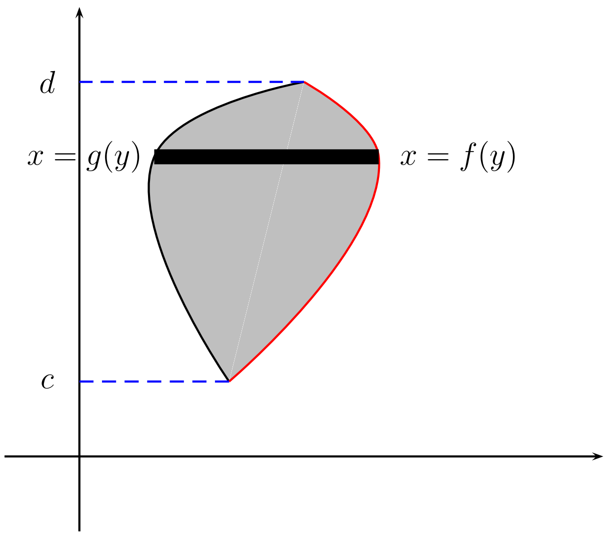

FIGURE 8. area between $x = f(y)$ and $x = g(y)$

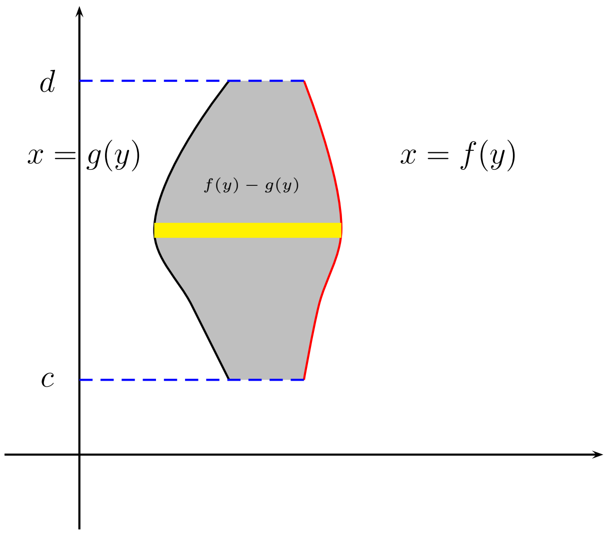

FIGURE 9. area between $x = f(y)$ and $x = g(y)$

Definition 0.3. [Area between $x = f(y)$ and $x = g(y)$:] If f and g are continuous functions on the interval $[c, d]$, and if $f(y) \geq g(y)$ for all y in $[c, d]$, then the ***Area*** of the region bounded above by $x = f(y)$, below by $x = g(y)$, on the left by the line $y = c$, and on the right by $y = d$ [see figures8 and 9] is

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

We usually use the notation

$$A = \int_c^d [x_{\text{Right}} - x_{\text{Left}}]\, dy.$$

Example 0.5. Find the area of the region enclosed by $y^2 = x$ and $x - 2y = 3$.

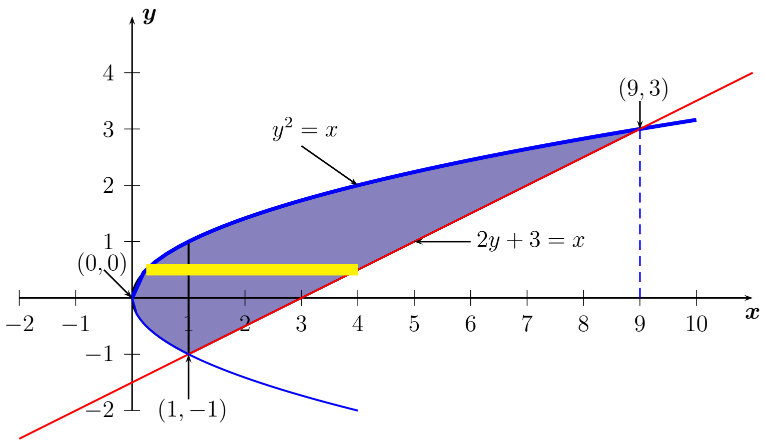

FIGURE 10

Solution: We start with solving the equations $y^2 = x$ and $x - 2y = 3$.

$$y^2 - 2y = 3$$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0$$

$$y = 3, -1.$$

Now, it is clear that y moves from -1 to 3 and we have $x = 2y + 3$ as the right curve and $x = y^2$ as the left curve. Hence

$$\begin{aligned}
A &= \int_c^d [f(y) - g(y)]\, dy \\
&= \int_{-1}^{3} \left[2y + 3 - y^2\right]\, dy \\
&= \left[y^2 + 3y - \frac{1}{3}y^3\right]_{-1}^{3} \\
&= \left[(9 + 9 - 9) - \left(1 - 3 + \frac{1}{3}\right)\right] \\
&= \left[9 - 1 + 3 - \frac{1}{3}\right] \\
&= \left[11 - \frac{1}{3}\right] \\
&= \frac{32}{3}.
\end{aligned}$$

Exercises 0.1. *In Exercises* $1-10$ *sketch the region bounded by the given curves and find the area of the region.*

(1) $y=x^2+3,\ y=x,\ x=-1,\ x=1$

(2) $y=x^2,\ y=x$

(3) $x+y^2=0,\ x=y^2+1,\ y=0,\ y=3$

(4) $y=x^2,\ y^2=x$

(5) $y=\sqrt{x},\ y=\dfrac{x}{2}$

(6) $y=x^2+1,\ y=3-x^2,\ x=-2,\ x=2$

(7) $y+x=0,\ y^2+x=2$

(8) $y=2x-x^2,\ y=x^3$

(9) $y=\dfrac{1}{x},\ y=\dfrac{1}{x^2},\ x=1,\ x=2$

(10) $y=\dfrac{1}{x},\ y=1,\ x=0,\ y=2$

In Exercises $11-15$ *find the area of the region bounded by the given curves by two methods (a) integrating with respect to* x*, and (b) integrating with respect to* y *.*

(11) $y^2+4x=0,\ y=2x+4$

(12) $y=\sqrt{x},\ y=-x,\ x=1,\ x=4$

(13) $y^2=x,\ 2y^2=x+4$

(14) $y=1-x^2,\ y=x-1$

(15) $y=x^2,\ y=4$