Improper Integrals

Definition 1: The definite integral $\int_a^b f(x)dx$ is called an improper integral if

1) At least one of the limits of integration is infinite, or
2) The integrand $f(x)$ has one or more points of discontinuity on the interval $[a, b]$.

Type I Improper Integral

a) If $\int_a^t f(x)dx$ exists for every number $t \ge a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx,$$

provided this limit exists (as a finite number).

b) If $\int_t^b f(x)dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called ***convergent*** if the corresponding limit exists and ***divergent*** if the limit does not exist.

c) If $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are convergent, then we defined

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

In part c) any number a can be used.

Example 1: Determine whether each integral is convergent or divergent

$$\int_0^\infty xe^{-x}dx \, .$$

Solution:

$$\int_0^\infty xe^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$$

$$= \lim_{t\to\infty} -xe^{-x} - e^{-x}\Big|_0^t = \lim_{t\to\infty}[-te^{-t} - e^{-t} + 0 + e^{-0}]$$

$$= \lim_{t\to\infty}[-te^{-t} - \frac{1}{e^t} + 1]$$

Note: $\lim_{t\to\infty}(-te^{-t}) \to \frac{-\infty}{\infty}$ (Indeterminate Form, use L'Hôpital's Rule).

$$\lim_{t\to\infty}(-te^{-t}) = \lim_{t\to\infty}(\frac{-t}{e^t}) = \lim_{t\to\infty}(\frac{-1}{e^t}) = 0..$$

Thus,

$$\int_0^\infty xe^{-x}dx = \lim_{t\to\infty}[-te^{-t} - \frac{1}{e^t} + 1]$$

$$= \lim_{t\to\infty}[-te^{-t} - \frac{1}{e^t} + 1] = (0 - 0 + 1) = 1.$$

Hence, $\int_0^\infty xe^{-x}\,dx$ converge to 1.

Example 2: Evaluate

$$\int_{-\infty}^{1} e^{2x}\,dx .$$

Solution:

$$\int_{-\infty}^{1} e^{2x}\,dx = \lim_{t\to-\infty}\int_t^1 e^{2x}\,dx = \lim_{t\to-\infty}\left.\frac{e^{2x}}{2}\right|_t^1$$

$$= \lim_{t\to-\infty}\left(\frac{e^2}{2}-\frac{e^{2t}}{2}\right) = \frac{e^2}{2} - \lim_{t\to-\infty}\left(\frac{e^{2t}}{2}\right) = \frac{e^2}{2} - 0 = \frac{e^2}{2}.$$

Hence, $\int_{-\infty}^{1} e^{2x}\,dx$ converge to $\frac{e^2}{2}$.

Example 3: Determine whether each integral is convergent or divergent

$$\int_{-\infty}^{\infty} (2x^2 - x + 3)\,dx .$$

Solution:

$$\int_{-\infty}^{\infty} (2x^2 - x + 3)\,dx = \int_{-\infty}^{0} (2x^2 - x + 3)\,dx + \int_{0}^{\infty} (2x^2 - x + 3)\,dx$$

$$= \lim_{a\to-\infty}\int_a^0 (2x^2 - x + 3)\,dx + \lim_{b\to\infty}\int_0^b (2x^2 - x + 3)\,dx$$

$$= \lim_{a\to-\infty}\left.\frac{2x^3}{3} - \frac{x^2}{2} + 3x\right|_a^0 + \lim_{b\to\infty}\left.\frac{2x^3}{3} - \frac{x^2}{2} + 3x\right|_0^b = \infty.$$

Thus, $\int_{-\infty}^{\infty} (2x^2 - x + 3)\,dx$ diverges.

Type II Improper Integral

1) If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$

2) If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

3) If f is continuous on the interval $[a,b]$ except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In each case, if the limit exists, then the improper integral is said to ***converge***; otherwise, the improper integral ***diverges.*** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example 4: Determine whether the integral is convergent or divergent

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx .$$

Solution: The function is undefined at $x = 1$. Since $x = 1$ is an asymptote, the function has no maximum. We could define this integral as:

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx = \lim_{b\to 1^-}\int_0^b \sqrt{\frac{1+x}{1-x}}\, dx = \int \sqrt{\frac{1+x}{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{x}{\sqrt{1-x^2}}\, dx .$$

Let $u = 1-x^2$, then $du = -2x\ dx$. Then

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx = \sin^{-1} x - \frac{1}{2}\int u^{-\frac{1}{2}} du = \sin^{-1} x - u^{\frac{1}{2}} = \lim_{b\to 1^-}\ \sin^{-1} x - \sqrt{1-x^2}\Big|_0^b$$

$$= \lim_{b\to 1^-}\ (\sin^{-1} b - \sqrt{1-b^2}) - (\sin^{-1} 0 - \sqrt{1}) = \frac{\pi}{2} + 1.$$

Thus, $\int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx$ converge to $\frac{\pi}{2}+1$.

Example 5: Determine whether the integral is convergent or divergent

$$\int_{\pi/4}^{\pi/2} \sec^2(x)\, dx .$$

Solution:

$$\int_{\pi/4}^{\pi/2} \sec^2(x)\, dx = \lim_{b\to(\pi/2)^-}\int_{\pi/4}^{b} \sec^2(x)\, dx$$

$$= \lim_{b\to(\pi/2)^-} \tan(x)\Big|_{\pi/4}^{b} = \lim_{b\to(\pi/2)^-} \left[\tan(b) - 1\right] = \infty$$

Thus, $\int_{\pi/4}^{\pi/2} \sec^2(x)\, dx$ diverges.

Example 6: Determine whether the integral is convergent or divergent

$$\int_0^4 \frac{1}{x^2+x-6}\, dx .$$

Solution:

$$\int_0^4 \frac{1}{x^2+x-6}\, dx = \int_0^4 \frac{1}{(x-2)(x+3)}\, dx$$

$$= \lim_{s\to 2^-}\int_0^s \frac{1}{(x-2)(x+3)}\, dx + \lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)(x+3)}\, dx$$

Now, $\frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$. Then $1 = A(x+3) + B(x-2)$, so $A = 1/5,\ B = -1/5$.

$$\int_0^4 \frac{1}{x^2+x-6}\, dx = \lim_{s\to 2^-}\int_0^s \left[\frac{1/5}{(x-2)} - \frac{1/5}{(x+3)}\right] dx + \lim_{t\to 2^+}\int_t^4 \left[\frac{1/5}{(x-2)} - \frac{1/5}{(x+3)}\right] dx$$

$$= \lim_{s\to 2^-}\left(\frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3|\Big|_0^s\right) + \lim_{t\to 2^+}\left(\frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3|\Big|_t^4\right)$$

Now, $\lim_{s\to 2^-}\left[\frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3|\right]_0^s = \lim_{s\to 2^-}\left[\frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|s+3| - \frac{1}{5}\ln(2) + \frac{1}{5}\ln(3)\right] = -\infty.$

Thus, $\int_0^4 \frac{1}{x^2+x-6}\, dx$ diverges.

A Comparison Test for Improper integrals

We use the Comparison Theorem at times when its impossible to find the exact value of an improper integral.

Theorem 1: Suppose that *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent,
2) If $\int_a^\infty f(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent

Note: $\int_1^\infty \frac{1}{x^p}$ is converge if $p > 1$ and diverge if $p \leq 1$.

***Example 7*:.** Use the Comparison Theorem to determine whether the integral is convergent or divergent.

$$\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$$

Solution: Observe that $\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} > \frac{1}{\sqrt{x}}$ on $[1,\infty)$. Now

$$\int_1^\infty \frac{1}{\sqrt{x}}dx = \int_1^\infty \frac{1}{x^{1/2}}dx \Rightarrow p = \frac{1}{2} < 1.$$

Then $\int_1^\infty \frac{1}{\sqrt{x}}dx$ is diverges (from above note). Thus, $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$ is divergent by Comparison Theorem.

Example 8: Use the Comparison Theorem to determine whether the integral is convergent or divergent.

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx$$

Solution: Observe that $e^{-x} \leq 1 \Rightarrow \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $0 < x \leq 1$. Now

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+} 2x^{1/2}\Big|_a^1 = \lim_{a\to 0^+}\left[2-2\sqrt{a}\right] = 2.$$

Thus, $\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx$ is convergent by Comparison Theorem.

Problems:

1- 18. In the following problems determine whether or not each of the improper integrals is convergent, and compute its value if it is.

		Answer
1.	$\int_2^{\infty} \frac{1}{x^2} dx$	$\frac{1}{2}$
2.	$\int_0^{\infty} e^{-x}\, dx$	1
3.	$\int_0^{\infty} \frac{x}{\sqrt{x}+x^2} dx$	Diverge
4.	$\int_0^{\infty} \frac{1}{x+2} dx$	Diverge
5.	$\int_{-\infty}^{-2} \frac{1}{x^2} dx$	$\frac{1}{2}$
6.	$\int_{-\infty}^{0} \frac{1}{\sqrt{1-x}} dx$	Diverge
7.	$\int_2^{\infty} \frac{1}{(x \ln x)} dx$	$\frac{1}{\ln 2}$
8.	$\int_1^{4} \frac{x}{\sqrt{16-x^2}} dx$	$\sqrt{15}$
9.	$\int_0^{a} \frac{dx}{\sqrt{a^2-x^2}}$	$\frac{\pi}{2}$
10.	$\int_0^{2} \frac{1}{x^2} dx$	
11.	$\int_0^{2} \frac{dx}{x^2-4}$	
12.	$\int_{-2}^{0} \frac{dx}{x+2}$	
13.	$\int_0^{\pi/2} \sec x\, dx$	
14.	$\int_{-1}^{1} \frac{dx}{x^2}$	
15.	$\int_0^{3} \frac{dx}{(x-1)^{2/3}}$	
16.	$\int_0^{\infty} \frac{dx}{1+x^2}$	$\frac{\pi}{2}$
17.	$\int_0^{\infty} e^{-x} \sin x\, dx$	
18.	$\int_0^{\infty} \frac{x^2}{1+x^2} dx$	

