Integration by Parts

I. Basic Technique

By the **Product Rule for Derivatives**,

$$\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x).$$

Thus,

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)dx = f(x)g(x)$$

So,

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}.$$

This formula for integration by parts often makes it possible to reduce a complicated integral involving a product to a simpler integral. By letting

$u = f(x)$	$dv = g'(x)dx$
$du = f'(x)dx$	$v = g(x)$

we get the more common formula for integration by parts:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Example 1: Find $\int x \ln x dx$.

Solution: Let

$u = \ln x$	$dv = xdx$
$du = \frac{1}{x}\,dx$	$v = \int xdx = \frac{1}{2}x^2$

Thus,

$$\int x\ln x dx = \int (\ln x)\,xdx = \int udv = uv - \int vdu = (\ln x)(\frac{1}{2}x^2) - \int (\frac{1}{2}x^2)(\frac{1}{x}dx)$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}\,x^2\ln x - \frac{1}{2}(\frac{1}{2}x^2) + C$$

$$= \frac{1}{2}\,x^2\ln x - \frac{1}{4}\,x^2 + C.$$

It is possible that when you set up an integral using integration by parts, the resulting integral will be more complicated than the original integral. In this case, change your substitutions for *u* and *dv*.

Example 2: Find $\int x\cos x\,dx$.

Solution: Let

$u = x$	$dv = \cos x\,dx$
$du = dx$	$v = \int \cos x\,dx = \sin x$

Thus,

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du = (x)(\sin x) - \int (\sin x)dx$$
$$= -x\sin x - \int \sin x dx = x\sin x + \cos x + C.$$

Example 3: Find $\int \ln x\,dx$.

Solution: Let

$u = \ln x$	$dv = 1dx$
$du = \frac{1}{x}dx$	$v = \int 1\ dx = x$

Thus,

$$\int \ln x\quad dx = \int (\ln x)(1\ \ dx) = \int u\,dv = uv - \int v\,du = (\ln x)(x) - \int (x)(\frac{1}{x}\ \ dx)$$
$$= x\ln x - \int 1\ \ dx = x\ln x - x + C.$$

Example 4: Find $\int \arctan x\,dx$.

Solution: Let

$u = \arctan x$	$dv = 1dx$
$du = \frac{1}{1+x^2}dx$	$v = \int 1\ dx = x$

Thus,

$$\int \arctan x\,dx = \int u\,dv = uv - \int v\,du$$
$$= x\arctan x - \int x\,(\frac{1}{1+x^2}dx) = x\arctan x - \int \frac{x}{1+x^2}dx$$
$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln\left|\ x^2+1\ \right| + C$$
$$= x\arctan x - \frac{1}{2}\ln(x^2+1) + C.$$

The next illustration of repeated integration by parts deserves special attention.

Example 5: Find $\int e^x \sin x\quad dx$.

Solution: Let

$u = e^x$	$dv = \sin x\,dx$
$du = e^x dx$	$v = \int \sin x dx = -\cos x$

Thus,

$$I = \int e^x \sin x\quad dx = (e^x)(-\cos x) - \int (-\cos x)(e^x dx) = -e^x\cos x + \int e^x \cos x dx$$

Notice that integration by parts is now needed to evaluate $\int e^x \cos x dx$.

$u = e^x$	$dv = \cos x\,dx$
$du = e^x dx$	$v = \int \cos x dx = \sin x$

Thus,

$$\int e^x \cos x dx = e^x \sin x - \int (\sin x)(e^x dx) = e^x \sin x - \int e^x \sin x dx = e^x \sin x - I.$$

Returning to the original problem,

$$I = \int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + e^x \sin x - I$$

$$2I = 2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x\, dx = \frac{1}{2}[-e^x \cos x + e^x \sin x] + C.$$

The next example illustrates an interesting type of integral that surprisingly requires integration by parts.

Example 6: $\int \sin\sqrt{x}\, dx$.

Solution: Let $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2udu = dx$.

$$\int \sin\sqrt{x}\, dx = \int (\sin u)(2udu) = 2\int u \sin u\, du$$

In example 2, we got the following using integration by parts:

$$\int u \sin u\, du = -u \cos u + \sin u + C$$

Thus,

$$\int \sqrt{x} \sin\sqrt{x}\, du = -2[\sqrt{x} \cos\sqrt{x} + \sin\sqrt{x}] + C .$$

In general, to evaluate $\int f(\sqrt[n]{x})\, dx$, let $u = \sqrt[n]{x} \Rightarrow u^n = x \Rightarrow nu^{n-1}du = dx$. Then

$$\int f(\sqrt[n]{x}) dx = n\int u^{n-1} f(u) du .$$

II. Tabular Integration

Integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, can be evaluated using tabular integration.

Example 7: Find $\int x^3 e^{2x} dx$.

Solution:

$f(x)$ and its derivatives		$g(x)$ and its antiderivatives
x^3	+	e^{2x}
$3x^2$	-	$e^{2x}/2$
$6x$	+	$e^{2x}/4$
6	-	$e^{2x}/8$
0	+	$e^{2x}/16$

$$\int x^3 e^{2x} dx = +\frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{6}{8}x e^{2x} - \frac{6}{16}e^{2x} + C = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C.$$

III. Reduction Formulas

Integration by parts can be used to derive **reduction formulas** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function.

Example 8: Prove the reduction formula $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$ and use the result to find $\int (\ln x)^2 dx$.

Solution: Let

$u = (\ln x)^n$	$dv = dx$
$du = n(\ln x)^{n-1}\frac{1}{x}dx$	$v = x$

Thus,

$$\int (\ln x)^n dx = x(\ln x)^n - \int (x)n(\ln x)^{n-1}(\frac{1}{x}\ dx) = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

Thus,

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\, dx = x(\ln x)^2 - 2[x(\ln x) - \int 1 dx] = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Example 9: Prove the reduction formula $\int (\sin x)^n dx = -\frac{1}{n}(\sin x)^{n-1}\cos x + \frac{n-1}{n}\int (\sin x)^{n-2} dx$ and use the result to find $\int \sin^3 x\, dx$.

Solution: Let

$u = (\sin x)^{n-1}$	$dv = \sin x \quad dx$
$du = (n-1)(\sin x)^{n-2}\cos x \quad dx$	$v = -\cos x$

Thus,

$$\begin{aligned}\int (\sin x)^n dx &= \int (\sin x)^{n-1}(\sin x\, dx) \\ &= (\sin x)^{n-1}(-\cos x) - \int (-\cos x)(n-1)(\sin x)^{n-2}\cos x\, dx \\ &= -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2}\cos^2 x\, dx \\ &= -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2}(1-\sin^2 x) dx \\ &= -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2} dx - (n-1)\int (\sin x)^n dx\end{aligned}$$

Therefore,

$$(n-1)\int (\sin x)^n dx + \int (\sin x)^n dx = -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2} dx$$

$$\int (\sin x)^n dx = -\frac{1}{n}(\sin x)^{n-1}\cos x + \frac{n-1}{n}\int (\sin x)^{n-2} dx .$$

$$\int \sin^3 x\, dx = -\frac{1}{3}(\sin x)^2\cos x + \frac{2}{3}\int \sin x\, dx = -\frac{1}{3}(\sin x)^2\cos x - \frac{2}{3}\cos x + C.$$

Problems:

1. Find an appropriate choice of u and dv for integration by parts of each integral.. Do not evaluate the integral.

a) $\int xe^{x}\,dx\,;\, u = -----,\ dv = -----$

b) $\int (x+1)\cos 7x\,dx\,;\, u = -----,\ dv = -----$

c) $\int \frac{x\,dx}{\sqrt{x+2}}\,;\, u = -----,\ dv = -----$

d) $\int \cos^{-1} x\,dx\,;\, u = -----,\ dv = -----$

2-15 Evaluate the integral

2. $\int x\cos 4x\,dx$	3. $\int y\sinh y\,dy$
4. $\int \sqrt[3]{x}\ln x\,dx$	5. $\int \arcsin x\,dx$
6. $\int \cos\sqrt{x}\,dx$	7. $\int \frac{\ln x}{x^{3}}\,dx$
8. $\int x\ \mathrm{arcsec} x\,dx$	9. $\int e^{x}\cos x\,dx$
10. $\int x^{3}\sin x^{2}dx$	11. $\int e^{\sqrt{x}}dx$
12. $\int x^{2}e^{3x}dx$	13. $\int x^{2}\sin x\,dx$
14. $\int e^{\sqrt[3]{x}}dx$	15. $\int \sin(\ln x)dx$

16. Prove the following reduction formula:

$$\int x^{m}(\ln x)^{n}\ dx = \frac{1}{m+1}\ x^{m+1}(\ln x)^{n} - \frac{n}{m+1}\int x^{m}(\ln x)^{n-1}dx\,,$$

for $m \neq -1$.