# Indeterminate Forms and L'Hôpital's Rule

## § 1. Indeterminate form type ( $0/0, \infty/\infty, -\infty/\infty$ ).

**Theorem** (*L'Hospital's Rule (L.R.)*). Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ near $a$ (except possibly at $a$). Suppose that $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, or that $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$. ( In other words, we have an indeterminate form type ( $0/0, \infty/\infty, -\infty/\infty$ ). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right side exists or is ( $\infty$ or $-\infty$ ).

***Note:*** In the rule " $x \to a$ " can be replaced by any of the following symbols

" $x \to a^+$ ", " $x \to a^-$ " , " $x \to \infty$ " , " $x \to -\infty$ " .

***Example 1***: Find $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

***Solution***: since $\lim\limits_{x \to 2} x^2 - 4 = 0$ and $\lim\limits_{x \to 2} x - 2 = 0$, then $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ gives us indeterminate form type $0/0$. We can apply (L.R.)

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \overset{\text{L.R.}}{=} \lim_{x \to 2} \frac{2x}{1} = \frac{2}{1} = 2.$$

***Example 2***: Find the limit $\lim\limits_{x \to \infty} \dfrac{x^3 + 1}{x^2 - 1}$.

***Solution***: since $\lim\limits_{x \to \infty} x^3 + 1 = \infty$ and $\lim\limits_{x \to \infty} x^2 - 1 = \infty$, then $\lim\limits_{x \to \infty} \dfrac{x^3 + 1}{x^2 - 1}$ gives us indeterminate form type $\infty/\infty$. Now, apply (L.R.) and simplify we get

$$\lim_{x \to \infty} \frac{x^3 + 1}{x^2 - 1} \overset{\text{L.R.}}{=} \lim_{x \to \infty} \frac{3x^2}{2x} = \lim_{x \to \infty} \frac{3x}{2} = \infty.$$

***Example 3***: Find the limit $\lim\limits_{x \to \infty} \dfrac{\ln x}{\sqrt[3]{x}}$.

***Solution***: since $\lim\limits_{x \to \infty} \ln x = \infty$ and $\lim\limits_{x \to \infty} \sqrt[3]{x} = \infty$, then $\lim\limits_{x \to \infty} \dfrac{\ln x}{\sqrt[3]{x}}$ gives us indeterminate form type $\infty/\infty$. Now, apply (L.R.) and simplify before using (L.R.) again we get

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} \overset{\text{L.R.}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{3} x^{\frac{-2}{3}}} = \lim_{x \to \infty} \frac{3}{x^{\frac{1}{3}}} = 0.$$

***Example 4***: Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x + 1}$.

**Solution:** Since $\lim\limits_{x\to 1} x^2-1=0$ and $\lim\limits_{x\to 1} x+1=2$, then $\lim\limits_{x\to 1}\dfrac{x^2-1}{x+1}$ don't give us indeterminate form. So no need to use (L.R.) and $\lim\limits_{x\to 1}\dfrac{x^2-1}{x+1}=\dfrac{1-1}{1+1}=\dfrac{0}{2}=0$.

If we try to use (L.R.) we will get a wrong answer $\lim\limits_{x\to 1}\dfrac{x^2-1}{x+1}=\lim\limits_{x\to 1}\dfrac{2x}{1}=2$.

## § 2. **Indeterminate form type ($0.\infty$).**

If $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=+\infty\ (-\infty)$, then $\lim\limits_{x\to a} f(x)g(x)$ called indeterminate form type $0.\infty$. We write

$$f\ g=\frac{f}{1/g}\ \text{,or}\ f\ g=\frac{g}{1/f}$$

and use L'Hôpital's Rule.

***Example 5*: Find** $\lim\limits_{x\to 0^+} x\ln x$.

***Solution*:** Since $\lim\limits_{x\to 0^+} x=0$ and $\lim\limits_{x\to 0^+}\ln x=-\infty$, the limit $\lim\limits_{x\to 0^+} x\ln x$ is an indeterminate form type $0.\infty$.
We will convert it to the form $\infty/\infty$ and apply (L.R.) as follows:

$$\lim_{x\to 0^+} x\ \ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\overset{\text{L.R.}}{=}\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}(-x)=0.$$

***Example 6*: Find** $\lim\limits_{x\to \frac{p}{2}^+}(x-\dfrac{p}{2})\tan x$.

***Solution*:** Since $\lim\limits_{x\to \frac{p}{2}^+} x-\dfrac{p}{2}=0$ and $\lim\limits_{x\to \frac{p}{2}^+}\tan x=\infty$, $\lim\limits_{x\to \frac{p}{2}^+}(x-\dfrac{p}{2})\tan x$ is an indeterminate form type $0.\infty$.
We will convert it to the form $0/0$ and apply (L.R.) as follows:

$$\lim_{x\to \frac{p}{2}^+}(x-\frac{p}{2})\tan x=\lim_{x\to \frac{p}{2}^+}\frac{x-\frac{p}{2}}{\frac{1}{\tan x}}=\lim_{x\to \frac{p}{2}^+}\frac{x-\frac{p}{2}}{\cot x}\overset{\text{L.R.}}{=}\lim_{x\to \frac{p}{2}^+}\frac{1}{-\csc^2 x}=-1.$$

## § 3. Indeterminate Difference.

If $\lim\limits_{x\to a} f(x)=\infty$ and $\lim\limits_{x\to a} g(x)=\infty$, then we say $[f(x)-g(x)]$ has the indeterminate form type ($\infty-\infty$).
To evaluate $\lim\limits_{x\to a}[f(x)-g(x)]$ we try by algebraic manipulation to convert it into a form of type $0/0, \infty/\infty, -\infty/\infty$, so that it may be possible to apply (L.R.)

***Example 7*: Find** $\lim\limits_{x\to 0^+}(\dfrac{1}{x}-\dfrac{1}{\sin x})$.

***Solution*:**

$$\lim_{x\to 0^+}(\frac{1}{x}-\frac{1}{\sin x})\overset{[\text{form }\infty-\infty]}{=}\lim_{x\to 0^+}(\frac{\sin x-x}{x\sin x})\qquad \text{(form } 0/0)$$

$$\overset{\text{L.R.}}{=} \lim_{x\to 0^+}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \qquad \text{(form } 0/0)$$

$$\overset{\text{L.R.}}{=} \lim_{x\to 0^+}\left(\frac{-\sin x}{2\cos x - x\sin x}\right) = 0.$$

***Example 8***: Find $\lim_{x\to\infty}(x-\sqrt{x^2-1})$.

***Solution***:

$$\lim_{x\to\infty}(x-\sqrt{x^2-1}) \overset{[\text{form } \infty-\infty]}{=} \lim_{x\to\infty}(x-\sqrt{x^2(1-1/x)})$$

$$= \lim_{x\to\infty} x(1-\sqrt{(1-1/x)}) \qquad (\text{form } 0.\infty)$$

$$= \lim_{x\to\infty}\frac{1-\sqrt{1-1/x}}{1/x} \qquad (\text{form } 0/0)$$

$$= \lim_{x\to\infty}\frac{-1}{2\sqrt{1-1/x}} = \frac{-1}{2}.$$

## § 4. Indeterminate power.

Limits of the form $\lim_{x\to a}[f(x)]^{g(x)}$ give rise to indeterminate form of types $0^0, \infty^0, 1^\infty$. We get those kinds in the following cases:

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type $0^0$
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type $\infty^0$
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type $1^\infty$

All three types are treated by first introducing a dependent variable $y = [f(x)]^{g(x)}$ and apply ln for both sides to get

$$\ln y = g(x)\ln f(x) = \frac{\ln f(x)}{1/g(x)},$$

then use (L.R.) to find $\lim_{x\to a}\ln y$. Now, $\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a}\ln y}$.

***Example 9*** : Find $\lim_{x\to 0^+}(1+\sin x)^{\cot x}$.

***Solution***: Since $\lim_{x\to 0^+}(1+\sin x) = 1$ and $\lim_{x\to 0^+}\cot x = +\infty$, the limit $\lim_{x\to 0^+}(1+\sin x)^{\cot x}$ is an indeterminate form type $1^{+\infty}$. Let $y = (1+\sin x)^{\cot x}$ take the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{\cot x} = \cot x\ln(1+\sin x) = \frac{\ln(1+\sin x)}{\frac{1}{\cot x}} = \frac{\ln(1+\sin x)}{\tan x}$$

The limit $\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin x)}{\tan x}$ is an indeterminate form of type $0/0$, by (L.R.),

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin x)}{\tan x} \overset{\text{L.R.}}{=} \lim_{x\to 0^+}\frac{\frac{\cos x}{1+\sin x}}{\sec^2 x} = 1.$$

Therefore, $\lim_{x\to 0^+}(1+\sin x)^{\cot x} = e^1 = e$.

***Example 10***: Find $\lim_{x\to\infty}(1+\frac{1}{x})^x$ .

***Solution***: Since $\lim_{x\to\infty}(1+\frac{1}{x})=1$ and $\lim_{x\to\infty}x=\infty$ , $\lim_{x\to\infty}(1+\frac{1}{x})^x$ is an indeterminate form type $1^{+\infty}$. Let $y=(1+\frac{1}{x})^x$ take the natural logarithm of both sides:

$$\ln y=\ln(1+\frac{1}{x})^x=x\ln(1+\frac{1}{x})=\frac{\ln(1+\frac{1}{x})}{1/x}.$$

The limit $\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(1+\sin x)}{\tan x}$ is an indeterminate form of type $0/0$, by (L.R.),

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x}=\frac{\frac{\cancel{-1/x^2}}{1+\frac{1}{x}}}{\cancel{-1/x^2}}=\lim_{x\to\infty}(1+\frac{1}{x})=1.$$

Therefore, $\lim_{x\to\infty}(1+\frac{1}{x})^x=e$ .

## ***Problems:***

1-22. Find the following limits

1. $\lim_{x\to0}\frac{e^x-1}{x^3}$
2. $\lim_{x\to+\infty}\frac{x^{-4/3}}{\sin(1/x)}$
3. $\lim_{x\to0^-}\frac{\tan x}{x^2}$
4. $\lim_{x\to1}\frac{x^2-1}{x^3-1}$
5. $\lim_{x\to1}\frac{\ln x}{x-1}$
6. $\lim_{x\to0}\frac{\tan x-x}{x^3}$
7. $\lim_{x\to\infty}\frac{e^x}{x^2}$
8. $\lim_{x\to p^-}\frac{\sin x}{1-\cos x}$
9. $\lim_{x\to0^+}x^x$
10. $\lim_{x\to\infty}e^{-x}\ln x$
11. $\lim_{x\to0}\frac{e^x-1}{\sin x}$
12. $\lim_{x\to0^+}\frac{\cot x}{\ln x}$
13. $\lim_{x\to+\infty}\frac{x^{10}}{e^x}$
14. $\lim_{x\to+\infty}x\,e^{-x}$
15. $\lim_{x\to0^+}x^x$
16. $\lim_{x\to\infty}e^{-x}\ln x$
17. $\lim_{x\to\frac{p}{2}^-}(\sec x-\tan x)$
18. $\lim_{x\to0^+}(\sin x)^{\tan x}$
19. $\lim_{x\to0}(1-2x)^{\frac{1}{x}}$
20. $\lim_{x\to0}(\csc x-1/x)$
21. $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$
22. $\lim_{x\to0}(1-2x)^{\frac{1}{x}}$