Trigonometric Integrals

In this section, we introduce techniques for evaluating integrals of the form

$$\int \sin^m x \cos^n x\,dx \qquad \int \tan^m x \sec^n x\,dx \qquad \int \cot^m x \csc^n x\,dx$$

where either *m* or *n* is a nonnegative integer.

I. Integrating Powers of the Sine and Cosine Functions

A. Useful trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin 2x = 2\sin x\cos x$
3. $\sin^2 x = \dfrac{1-\cos 2x}{2}$
4. $\cos^2 x = \dfrac{1+\cos 2x}{2}$
5. $\sin x\cos y = \dfrac{1}{2}[\sin(x-y)+\sin(x+y)]$
6. $\sin x\sin y = \dfrac{1}{2}[\cos(x-y)-\cos(x+y)]$
7. $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
8. $\cos x\cos y = \dfrac{1}{2}[\cos(x-y)+\cos(x+y)]$

B. Reduction formulas

1. $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$
2. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

Example 1: Find $\int \sin^2 x\,dx$.

Solution: Method 1(Integration by parts):

$$\int \sin^2 x\,dx = \int \sin x\,(\sin x\,dx\,).$$

Let

$u = \sin x$	$dv = \sin x\,dx$
$du = \cos x\,dx$	$v = \int \sin x\,xd = -\cos x$

Thus,

$$\int \sin^2 x\,dx = (\sin x\,)(-\cos x\,) + \int \cos^2 x\,dx = -\sin x\cos x + \int (1-\sin^2 x\,)\,dx$$

$$= -\sin x\cos x + \int 1\,dx - \int \sin^2 x\,dx$$

$$= -\sin x\cos x + x - \int \sin^2 x\,dx$$

Therefore,

$$\int \sin^2 x\,dx + \int \sin^2 x\,dx = 2\int \sin^2 x\,dx = -\sin x\cos x + x + C$$

$$\Leftrightarrow \int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C.$$

Method 2 (Trig. identity): $\displaystyle\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x\,)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

Method 3 (Reduction formula): $\int \sin^2 x\, dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1 dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C.$

Example 2: Find $\int \cos^3 x\, dx$.

Solution: Use the reduction formula:

$$\begin{aligned}\int \cos^3 x\, dx &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\, dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C \\ &= \frac{1}{3}\sin x(1-\sin^2 x) + \frac{2}{3}\sin x + C = \sin x - \frac{1}{3}\sin^3 x + C.\end{aligned}$$

II Integrating Products of the Sines and Cosines

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosine. Then, expand and integrate.

$$\begin{aligned}\int \overbrace{\sin^{2k+1}}^{\text{odd}} x \cos^n x\, dx &= \int \overbrace{\left(\sin^2 x\right)^k}^{\text{convert to cosine}} \cos^n x\, \overbrace{\sin x\, dx}^{\text{save for } du} \\ &= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x\, dx\end{aligned}$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sine. Then, expand and integrate.

$$\begin{aligned}\int \sin^m x\, \overbrace{\cos^{2k+1}}^{\text{odd}} x\, dx &= \int \sin^m x\, \overbrace{\left(\cos^2 x\right)^k}^{\text{convert to sine}} \overbrace{\cos x\, dx}^{\text{save for } du} \\ &= \int \sin^m x \left(1-\sin^2 x\right)^k \sin x\, dx\end{aligned}$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities (1) and (2) to convert the integrand to odd powers of the cosine. Then, proceed as in case 2. It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

Example 3: Find $\int \sin^3 x \cos^2 x\, dx$

Solution:

$$\begin{aligned}\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \sin x \cos^2 x\, dx = \int (1-\cos^2 x)\cos^2 x \sin x\, dx = \\ &= \int (\cos^2 x - \cos^4 x)(\sin x\, dx)\end{aligned}$$

Let $u = \cos x \Rightarrow du = -\sin x\, dx$. Thus,

$$\begin{aligned}\int \sin^3 x \cos^2 x\, dx &= -\int (u^2 - u^4) du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.\end{aligned}$$

Example 4: Find $\int \sin^2 x \cos^2 x \, dx$.

Solution:

$$\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{4}\int (1-\cos^2 2x)\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 2x \, dx = \frac{1}{4}x - \frac{1}{4}\int \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos 4x \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

Example 5: Find $\int \sin 4x \cos 3x \, dx$.

Solution: Method 1 (Trig. identity):

$$\int \sin 4x \cos 3x \, dx = \frac{1}{2}\int (\sin x + \sin 7x) \, dx = -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C.$$

Method 2 (Integration by parts): Let

$u = \sin 4x$	$dv = \cos 3x \, dx$
$du = 4\cos 4x \, dx$	$v = \frac{1}{3}\sin 3x$

Thus,

$$\int \sin 4x \cos 3x \, dx = (\sin 4x)\left(\frac{1}{3}\sin 3x\right) - \frac{4}{3}\int \cos 4x \sin 3x \, dx = \frac{1}{3}\sin 4x \sin 3x - \frac{4}{3}\int \cos 4x \sin 3x \, dx.$$

Find $\int \cos 4x \sin 3x \, dx$

$u = \cos 4x$	$dv = \sin 3x \, dx$
$du = -4\sin 4x \, dx$	$v = -\frac{1}{3}\cos 3x$

Thus,

$$\int \cos 4x \sin 3x \, dx = -\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x \, dx \ .$$

Returning to the original integral,

$$\int \sin 4x \cos 3x \, dx = \frac{1}{3}\sin 4x \sin 3x - \frac{4}{3}\left\{-\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x \, dx\right\}$$

$$= \frac{1}{3}\sin 4x \sin 3x + \frac{4}{9}\cos 4x \cos 3x + \frac{16}{9}\int \sin 4x \cos 3x \, dx$$

Then

$$\int \sin 4x \cos 3x \, dx - \frac{16}{9}\int \sin 4x \cos 3x \, dx = \frac{1}{3}\sin 4x \sin 3x + \frac{4}{9}\cos 4x \cos 3x$$

$$-\frac{7}{9}\int \sin 4x \cos 3x \, dx = \frac{1}{3}\sin 4x \sin 3x + \frac{4}{9}\cos 4x \cos 3x$$

Thus

$$\int \sin 4x \cos 3x \, dx = -\frac{3}{7}\sin 4x \sin 3x - \frac{4}{7}\cos 4x \cos 3x + C.$$

III Integrating Powers of the Tangent and Secant Functions

A. Useful trigonometric identity: $\tan^2 x + 1 = \sec^2 x$

B. Useful integrals

1. $\int \sec x \tan x \, dx = \sec x + C$
2. $\int \sec^2 x \, dx = \tan x + C$
3. $\int \tan x dx = \ln|\sec x| + C$
4. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

C. Reduction formulas

1. $\int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x \, dx$
2. $\int \tan^n x \, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

Example 6: Find $\int \tan^2 x \, dx$.

Solution:

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \int \sec^2 x \, dx - \int 1 dx = \tan x - x + C.$$

Example 7: Find $\int \tan^3 x \, dx$.

Solution:

$$\int \tan^3 x \, dx = \frac{\tan^2 x}{2} - \int \tan x \, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

Example 8: Find $\int \sec^3 x \, dx$.

Solution:

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

IV Integrating Products of the Tangents and Secants

1. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \overbrace{\tan^{2k+1}}^{\text{odd}} x \sec^n x dx = \int \overbrace{\left(\tan^2 x\right)^k}^{\text{convert to secants}} \sec^{n-1} x \overbrace{\sec x \tan x dx}^{\text{save for } du}$$
$$= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x dx$$

2. If the power of the secant is even and positive, save a secant squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \tan^m x \overbrace{\sec^{2k}}^{\text{even}} x \, dx = \int \tan^m x \overbrace{(\sec^2 x)^{k-1}}^{\text{convert to tangents}} \overbrace{\sec^2 x \, dx}^{\text{save for } du}$$
$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent squared factor to secants; then expand and repeat if necessary.

$$\int \tan^m x\, dx = \int \tan^{m-2} x \overbrace{(\tan^2 x)}^{\text{convert to secant}} dx = \int \tan^{m-2} x\, (\sec^2 x - 1) dx$$
$$= \int \tan^{m-2} x\, (\sec^2 x)\, dx - \int \tan^{m-2} x\, dx$$

4. If the integral is of the form $\int \sec^n x\, dx$, where n is odd and positive, possibly use integration by parts.
5. If none of the first four cases apply, try converting to sines and cosines.

Example 9: Find $\int \tan x \sec^2 x\, dx$.

Solution: Let $u = \tan x \Rightarrow du = \sec^2 x dx$, then

$$\int \tan x \sec^2 x\, dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

Example 10: Find $\int \tan x \sec^4 x\, dx$.

Solution:

$$\int \tan x \sec^4 x\, dx = \int \tan x \sec^2 x \sec^2 x\, dx = \int \tan x\, (1 + \tan^2 x) \sec^2 x\, dx$$
$$= \int \tan x \sec^2 x \quad dx + \int \tan^3 x \sec^2 dx.$$

Let $u = \tan x \Rightarrow du = \sec^2 x dx$. Thus,

$$\int \tan x \sec^4 x\, dx = \int u du + \int u^3 du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.$$

Example 11: Find $\int \tan x \sec^3 x\, dx$.

Solution:

$$\int \tan x \sec^3 x\, dx = \int \sec^2 x\, (\sec x \tan x\, dx) \ .$$

Let $u = \sec x \Rightarrow du = \sec x \tan x\, dx$, thus

$$\int \tan x \sec^3 x\, dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C.$$

Example 12: Find $\int \tan^2 x \sec^3 x\, dx$.

Solution:

$$\int \tan^2 x \sec^3 x\, dx = \int (\sec^2 x - 1)\sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\ dx.$$

Using the reduction formula,

$$\int \sec^5 x\, dx = \frac{1}{4} \ \sec^3 \tan x + \frac{3}{4}\int \sec^3 x\, dx.$$

Thus,

$$\int \tan^2 x \sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x\, dx - \int \sec^3 x\, dx$$
$$= \frac{1}{4} \ \sec^3 x \tan x - \frac{1}{4}\overbrace{\int \sec^3 x\, dx}^{\text{From Example 7}}$$
$$= \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln\left|\sec x + \tan x\right| + C.$$

Example 13: Find $\int \sqrt{\tan x} \;\; \sec^4 x \, dx$.

Solution:

$$\int \sqrt{\tan x} \;\; \sec^4 x \, dx = \int \sqrt{\tan x} \sec^2 x \sec^2 x \, dx = \int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x \, dx.$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$, then

$$\int \sqrt{\tan x} \;\; \sec^4 x \, dx = \int \sqrt{\tan x} \sec^2 x \, dx + \int \sqrt{\tan x} \tan^2 x \sec^2 x \, dx$$

$$= \int u^{\frac{1}{2}} du + \int u^{\frac{5}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + \frac{2}{7} u^{\frac{7}{2}} + C$$

$$= \frac{2}{3}(\tan x)^{\frac{3}{2}} + \frac{2}{7}(\tan x)^{\frac{7}{2}} + C.$$

Example 14: Find $\int \sqrt{\sec x} \tan x \, dx$.

Solution: Let $u = \sqrt{\sec x} \Rightarrow u^2 = \sec x \Rightarrow 2u du = \sec x \tan x dx = u^2 \tan x dx$. So $\tan x \, dx = \frac{2u du}{u^2} = \frac{2}{u} du$.

Thus,

$$\int \sqrt{\sec x} \tan x \, dx = \int u \left(\frac{2}{u} du\right) = 2\int 1 du = 2u + C = 2\sqrt{\sec x} + C.$$

Example 15: Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$.

Solution:

$$\int \frac{\tan^3 x}{\sqrt{\sec x}} dx = \int \sec x^{-1/2} \tan^3 x dx$$

$$= \int \sec x^{-3/2} (\tan^2 x)(\sec x \tan x) dx$$

$$= \int \sec x^{-3/2} (\sec^2 x - 1)(\sec x \tan x) dx$$

$$= \int \left[\sec x^{1/2} - \sec x^{-3/2}\right](\sec x \tan x) dx$$

$$= \frac{2}{3} \sec x^{3/2} + 2 \sec x^{-1/2} + C.$$

Example 16: Find $\int_0^{\pi/4} \tan^4 x \;\, dx$.

Solution:

$$\int_0^{\pi/4} \tan^4 x dx = \int_0^{\pi/4} \tan^2 x \, (\tan^2 x) dx = \int_0^{\pi/4} \tan^2 x \;\, (\sec^2 x - 1) dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x dx - \int_0^{\pi/4} \tan^2 x dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x dx - \int_0^{\pi/4} (\sec^2 x - 1) dx = \frac{\tan^3 x}{3} - \tan x + x \Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{2}{3}.$$

IIV Integrals Involving Cotangent and Cosecant

NOTE: The guidelines for integrals involving cotangent and cosecant would be similar to that of integrals involving tangent and secant.

Example 17: Find $\int \csc^4 x \cot^4 x\,dx$

Solution:

$$\begin{aligned}\int \csc^4 x \cot^4 x\,dx &= \int \csc^2 x \cot^4 x\,(\csc^2 x)\,dx \\ &= \int (1+\cot^2 x)(\cot^4 x)(\csc^2 x)\,dx \\ &= -\int (\cot^4 x + \cot^6 x)(-\csc^2 x)\,dx \\ &= -\frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + C.\end{aligned}$$

Problems:

1. Prove the reduction formula: $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$
2. Prove the reduction formula: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$
3. Prove the reduction formula: $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$
4. Prove the reduction formula: $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

5. $\int \tan^3 3x\,dx$
6. $\int_0^{\pi/4} \cos^2(2x)\,dx$
7. $\int \tan^3 x \sec^3 x\,dx$
8. $\int \sqrt{\sin x}\cos^3 x\,dx$
9. $\int \cos^3 x \sin^2 x\,dx$
10. $\int \sin^2 x \cos^2 x\,dx$
11. $\int \tan^5 x \sec x\,dx$
12. $\int \cos^3 5\theta\,d\theta$
13. $\int \cos^3 x \csc^3 x\,dx$
14. $\int \cot^2(\frac{x}{3})\,dx$
15. $\int \cot x \csc^2 x\,dx$
16. $\int \sin^4 x \cos^4 x\,dx$
17. $\int \sin^6 x\,dx$
18. $\int \cos 5x \cos 7x\,dx - \int \sin 5x \sin 7x\,dx$
19. $\int \cos^2 x \sin^4 x\,dx$
20. $\int \cos^3 x \sin^{-5/2} x\,dx$