VOLUME-DISK AND WASHER METHODS

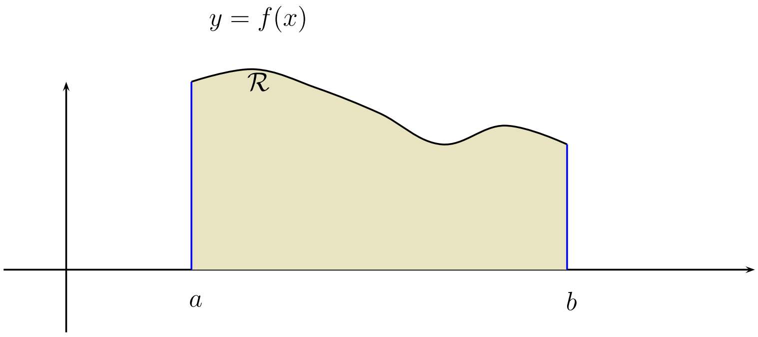

FIGURE 1

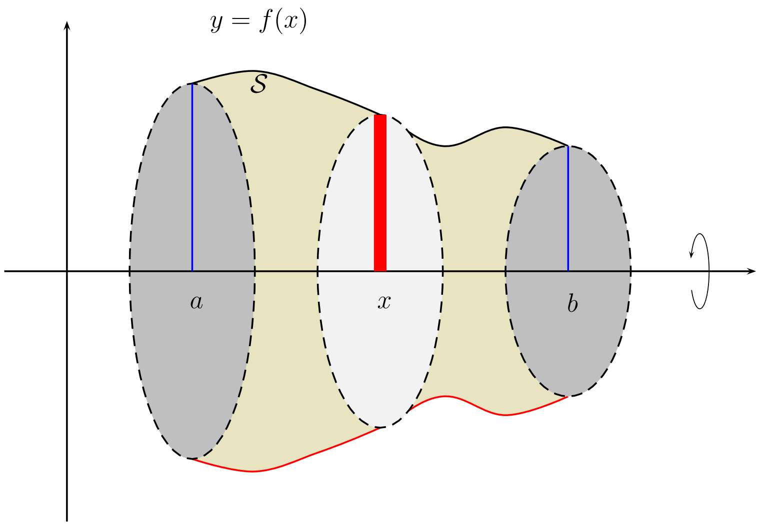

FIGURE 2

Definition 0.1. [**Volumes of solids of revolution: Disk method: Perpendicular to the x-axis**] Let $\mathcal{R}$ be the region bounded by $y = f(x),\ y = 0,\ x = a$, and $x = b$. When this region is revolved about x-axis we get a solid $\mathcal{S}$. A cross-section through x perpendicular to the $x-$axis is a circular disk with radius equal to $f(x)$. Hence the area of the cross-section $A(x) = \pi.(radius)^2 = \pi[f(x)]^2$. The ***Volume*** of $\mathcal{S}$ is given by the following formula

$$V = \int_a^b A(x)\ dx = \int_a^b \pi[f(x)]^2\ dx.$$

Example 0.1. find the volume of the solid that is obtained when the region $y = \sqrt{x},\ y = 0,\ x = 0$, and $x = 2$ is revolved about the x-axis

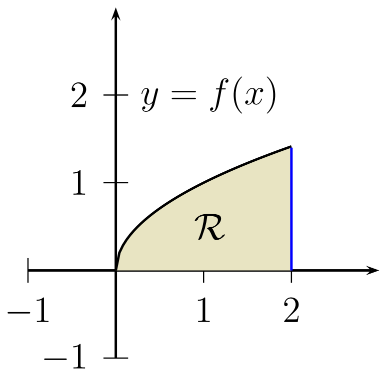

FIGURE 3

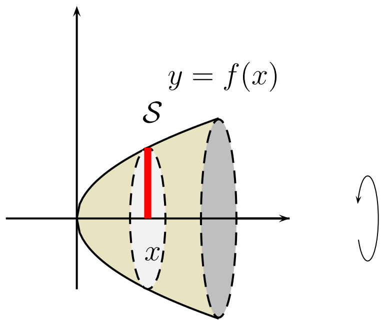

FIGURE 4

Solution:

Since we are rotating about the $x-$axis, we should integrate with respect to x. The cross-section at any $x \in [0, 2]$ perpendicular to the $x-$axis is a disk with radius $\sqrt{x}$. Then

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2 \, dx \\ &= \int_0^2 \pi[\sqrt{x}]^2 \, dx \\ &= \int_0^2 \pi[x] \, dx \\ &= \pi \left[\frac{1}{2}x^2\right]_0^2 \\ &= \pi \left[\frac{1}{2}4\right] \\ &= 2\pi. \end{aligned}$$

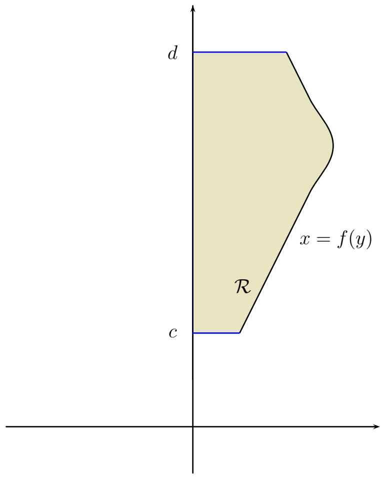

FIGURE 5

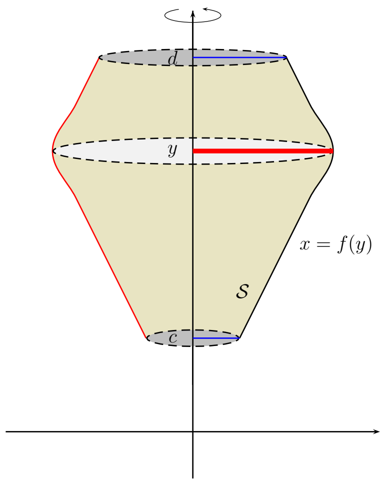

FIGURE 6

Definition 0.2. [**Volumes of solids of revolution: Disk method: Perpendicular to the y-axis**]
Let $\mathcal{R}$ be the region bounded by $x = f(y),\ x = 0,\ y = c$, and $y = d$. When this region is revolved about y-axis we get a solid $\mathcal{S}$. A cross-section through y perpendicular to the $y-$axis is a circular disk with radius equal to $f(y)$. Hence the area of the cross-section $A(y) = \pi.(radius)^2 = \pi[f(y)]^2$. The ***Volume*** of $\mathcal{S}$ is given by the following formula

$$V = \int_c^d A(y)\ dy = \int_c^d \pi[f(y)]^2\ dy.$$

Example 0.2. find the volume of the solid that is obtained when the region $y = x^3$, $x = 0$, and $y = 8$ is revolved about the y-axis

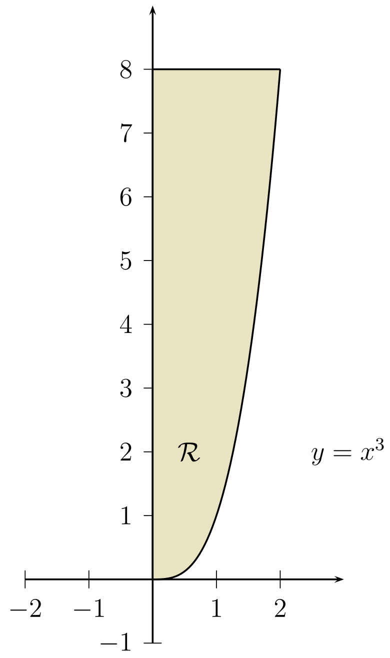

FIGURE 7

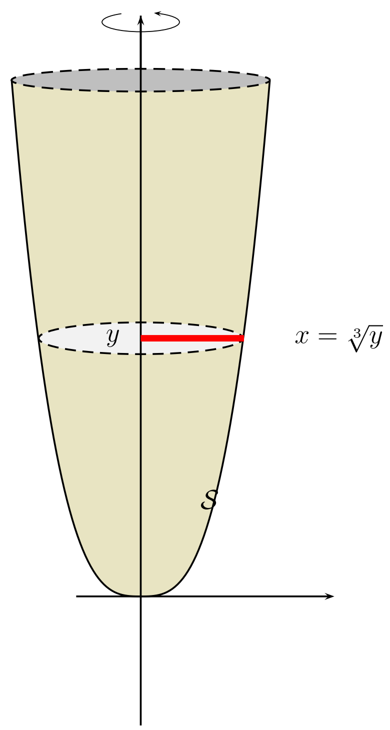

FIGURE 8

Solution:

Since we are rotating about the $y-$axis, we should integrate with respect to y. The cross-section at any $y \in [0, 8]$ perpendicular to the $y-$axis is a disk with radius $\sqrt[3]{y}$. Then

$$\begin{aligned}
V &= \int_c^d \pi[f(y)]^2 \; dy \\
&= \int_0^8 \pi[\sqrt[3]{y}]^2 \; dy \\
&= \int_0^8 \pi \; y^{\frac{2}{3}} \; dy \\
&= \pi \left[\frac{5}{3} y^{\frac{5}{3}}\right]_0^8 \\
&= \pi \left[\frac{5}{3} 8^{\frac{5}{3}}\right] \\
&= \frac{96\pi}{3}.
\end{aligned}$$

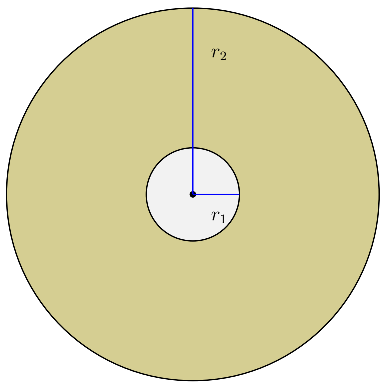

FIGURE 9

Note 0.1. *Note that the area of an annular [region bounded by two circles one of radius r_1 and the other of radius r_2 with $r_1 < r_2$.] is* $A = \pi[r_2^2 - r_1^2] = \pi[(\text{ outer radius })^2 - (\text{ inner radius })^2]$.

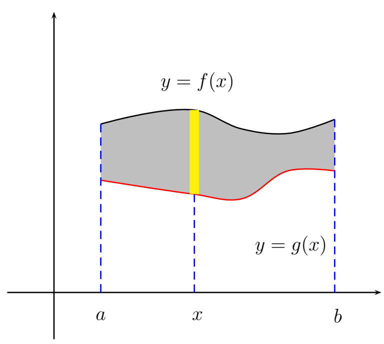

FIGURE 10

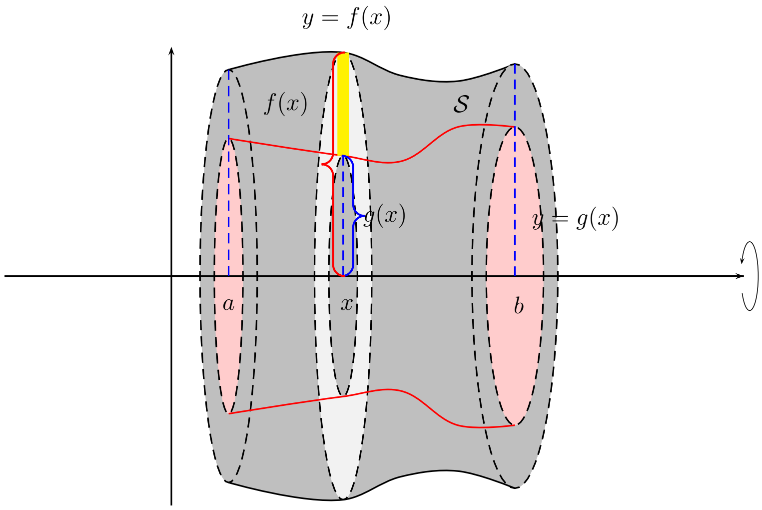

FIGURE 11

Definition 0.3. [**Volumes of solids of revolution: Washer method: Perpendicular to the x-axis**]

Let $\mathcal{R}$ be the region bounded by $y = f(x),\ y = g(x),\ x = a$, and $x = b$, where $f(x) \geq g(x),\ \forall\ x \in [a, b]$. When this region is revolved about x-axis we get a solid $\mathcal{S}$. A cross-section taken perpendicular to the $x-$axis at any point x is an annular "washer-shaped" with inner radius equal to $g(x)$ and outer radius equal to $f(x)$. Hence the area of the cross-section

$A(x) = \pi[(\text{ outer radius })^2 - (\text{ inner radius })^2] = \pi[[f(x)]^2 - [g(x)]^2]$.

Then the ***Volume*** of $\mathcal{S}$ is given by the following formula

$$V = \int_a^b A(x)\ dx = \int_a^b \pi[(f(x))^2 - (g(x))^2]\ dx.$$

A similar definition about the $y-$axis should be obtained by the reader.

Example 0.3. Find the volume of the solid obtained by revolved the region $y = x$ and $y = x^2$ about the $x-$axis

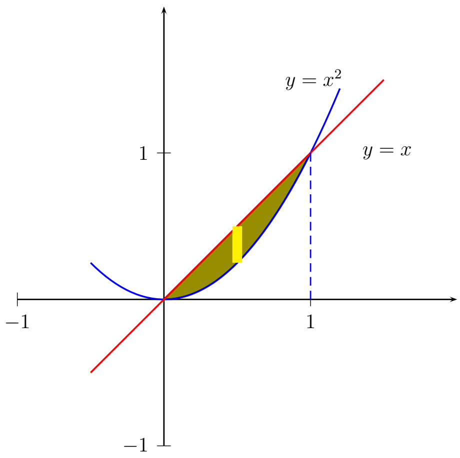

FIGURE 12

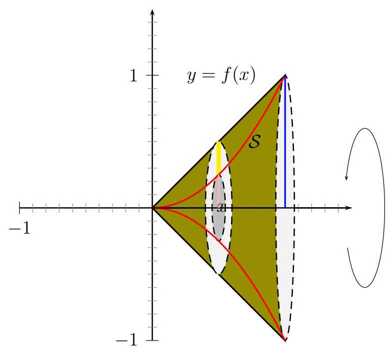

FIGURE 13

Solution:

Since we are rotating about the $x-$axis, we should integrate with respect to x. The cross-section at any $x \in [0, 1]$ perpendicular to the $x-$axis is an annular "washer-shaped" with inner radius equal to x^2 and outer radius equal to x. Then

$$\begin{aligned} V &= \int_a^b \pi[(f(x))^2 - (g(x))^2]\ dx \\ &= \int_0^1 \pi[(x)^2 - (x^2)^2]\ dx \\ &= \int_0^1 \pi\ [x^2 - x^4]\ dx \\ &= \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 \\ &= \pi \left[\frac{1}{3} - \frac{1}{5}\right] \\ &= \frac{2\pi}{15}. \end{aligned}$$

Example 0.4. Find the volume of the solid obtained by revolved the region $y = x$ and $y = x^2$ about the $y-$axis

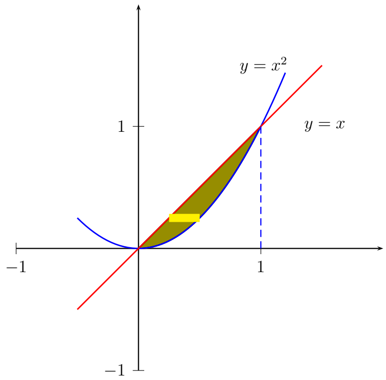

FIGURE 14

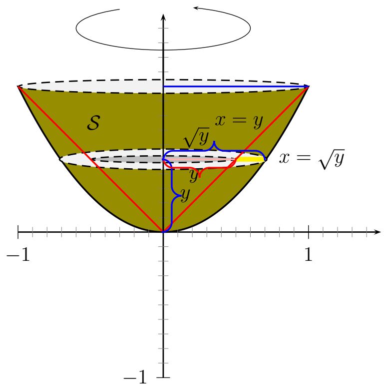

FIGURE 15

Solution:

Since we are rotating about the $y-$axis, we should integrate with respect to y. The cross-section at any $y \in [0, 1]$ perpendicular to the $y-$axis is an annular "washer-shaped" with inner radius equal to y and outer radius equal to $\sqrt{y}$. Then

$$
\begin{aligned}
V &= \int_c^d \pi[(\text{ outer radius })^2 - (\text{ inner radius})^2]\ dy \\
&= \int_0^1 \pi[(\sqrt{y})^2 - (y)^2]\ dy \\
&= \int_0^1 \pi\ [y - y^2]\ dx \\
&= \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 \\
&= \pi \left[\frac{1}{2} - \frac{1}{3}\right] \\
&= \frac{\pi}{6}.
\end{aligned}
$$

Exercises 0.1. *In Exercises* $1-10$ *find the volume of the solid obtained by rotating the region bounded by the given curves about the given axis. Sketch the region, the solid, and a typical disk or "washer".*

(1) $y=x^2,\ y=x,\ y=0,$ *about the* $x-$*axis*

(2) $y=x^2,\ y=4,\ x=0,\ x=2,$ *about the* $y-$*axis*

(3) $x+y=1,\ x=0,\ y=0,$ *about the* $x-$*axis*

(4) $x=y-y^2,\ x=0,$ *about the* $y-$*axis*

(5) $y=2x-x^2,\ y=0,\ x=0,\ x=1,$ *about the* $y-$*axis*

(6) $y=x^2+1,\ y=3-x^2,$ *about the* $x-$*axis*

(7) $2y-x=0,\ y^2-x=0,$ *about the* $y-$*axis*

(8) $y=\sqrt{x-1},\ y=0,\ x=5,$ *about the* $y-$*axis*

(9) $y=\cos x,\ y=\sin x,\ x=0,\ x=\frac{\pi}{4},$ *about the* $x-$*axis*

(10) $y=\frac{-1}{x},\ y=0,\ x=1,\ y=3,$ *about the* $x-$*axis*

