VOLUMES BY CYLINDRICAL SHELLS

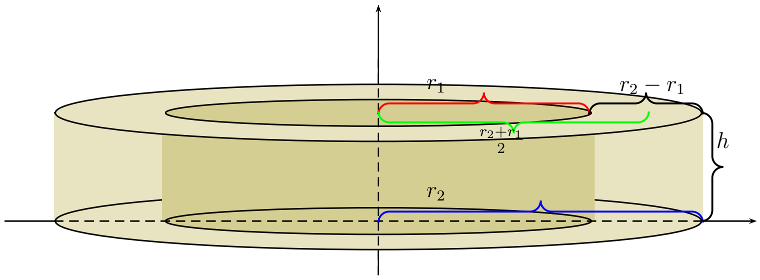

FIGURE 1

A ***Cylindrical shell*** is a solid enclosed by two concentric right-circular cylinders. The volume V of Cylindrical shell having inner radius r_1 , outer radius r_2 and height h can be written as

$$V = \pi r_2^2 h - \pi r_1^2 h$$

$$= \pi(r_2^2 - r_1^2)h$$

$$= \pi(r_2 + r_1)(r_2 - r_1)h$$

$$= 2\pi \tfrac{r_2+r_1}{2} h(r_2 - r_1)$$

$$= 2\pi r h \Delta r \qquad \text{where} \quad r = \tfrac{r_2+r_1}{2} \quad \text{and} \quad \Delta r = r_2 - r_1$$

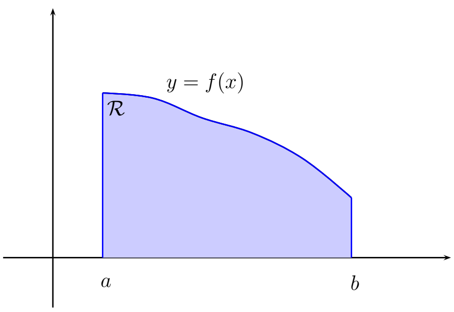

FIGURE 2

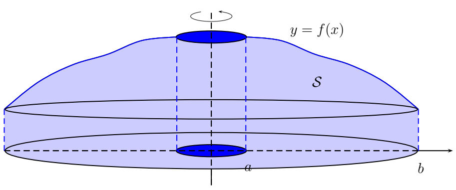

FIGURE 3

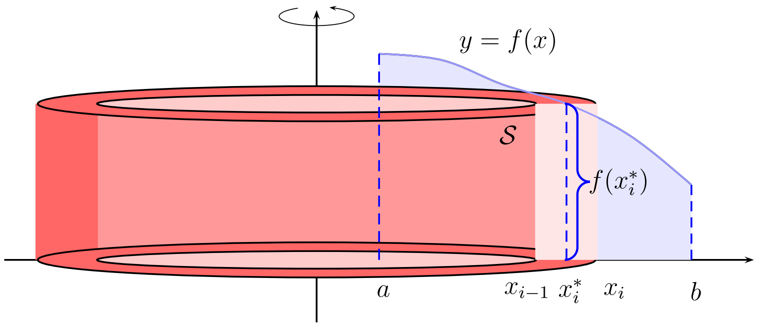

FIGURE 4

Let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x),\ y = 0,\ x = a,$ and $x = b$. We partition the interval $[a, b]$ into n subinterval $a = x_0 < x_1 < \dots < x_n = b$ with equal length $\triangle x$. Let x_i^* be the midpoint of $[x_{i-1}, x_i]$. If the rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$ is rotated about the $y-$axis we get a cylindrical shell with average radius x_i^*, height $f(x_i^*)$ and thickness $\triangle x_i = \triangle x$. Then its volume is $V_i = 2\pi x_i^* f(x_i^*)\Delta x$ and

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x)\, dx$$

Definition 0.1. Let S be the solid obtained by revolved the region $y = f(x),\ y = 0,\ x = a,\ x = b.$ about the $y-$axis .Then the volume of is

$$V = \int_a^b 2\pi x f(x)\, dx$$

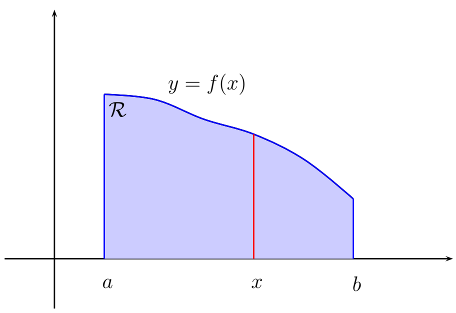

FIGURE 5

Note 0.1. *At any x in the interval $[a, b]$, the vertical line segment from the $x-$axis to the graph of $y = f(x)$ can be viewed as a cross-section of the region $\mathcal{R}$ at x.(see figure 5 and 6) When the region $\mathcal{R}$ is revolved about $y-$axis, the cross-section at x generates the surface of a right-circular cylinder of hight $f(x)$*

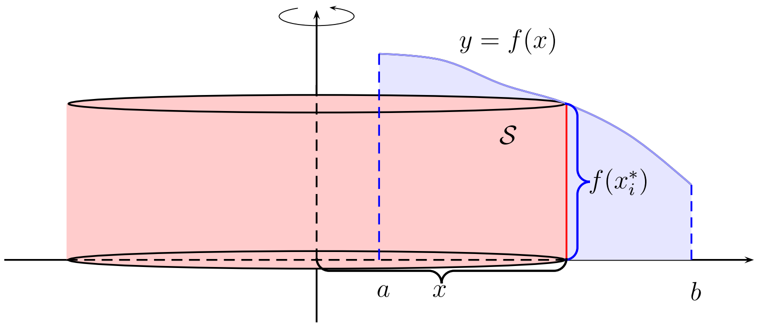

FIGURE 6

and radius x. The area of this surface is $2\pi x f(x)$. *So we can say that the volume by cylindrical shell is the integral of the surface area generated by any arbitrary cross-section of* $\mathcal{R}$ *taken parallel to rotation axis.*

$$V = \int_a^b 2\pi x f(x)\, dx$$

Example 0.1. find the volume of the solid obtained by rotating the region $y = 4x(1-x)$ and $y = 0$ about the $y-$axis

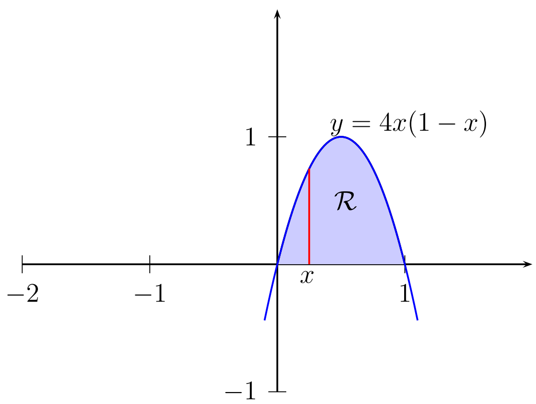

FIGURE 7

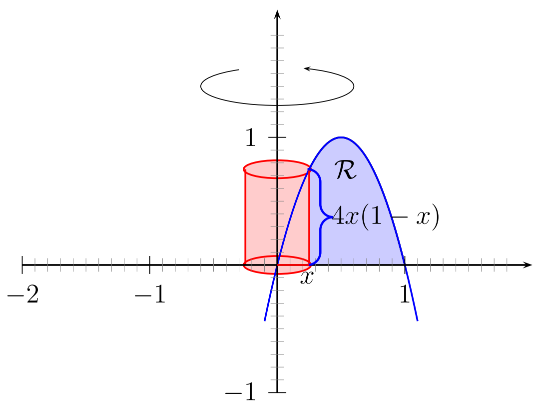

FIGURE 8

Solution: Since we will take a cross-section parallel to the $y-$axis, we will integrate with respect to x. By solving the equation $4x(1-x) = 0$, we get $x = 0,\ x = 1$. At each $x \in [0, 1]$, the cross-section of the region $\mathcal{R}$ parallel to $y-$axis generates a cylindrical surface of hight $4x(1-x)$ and radius x. Since the area of the surface is $2\pi x 4x(1-x)$, the volume of the solid is

$$\begin{aligned}
V &= \int_a^b 2\pi x f(x)\, dx \\
&= \int_0^1 8\pi x^2(1-x)\, dx \\
&= \int_0^1 8\pi x^2(1-x)\, dx \\
&= \int_0^1 8\pi x^2 - x^3\, dx \\
&= 8\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 \\
&= 8\pi \left[\frac{1}{3} - \frac{1}{4}\right] \\
&= 8\pi \frac{1}{12} \\
&= \frac{2\pi}{3}.
\end{aligned}$$

Example 0.2. find the volume of the solid obtained by rotating the region $y = \sqrt{x}$, $y = 0$, and $x = 2$ about the $x-$axis

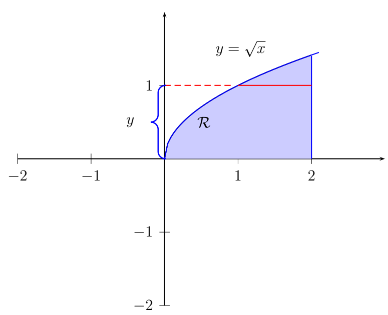

FIGURE 9

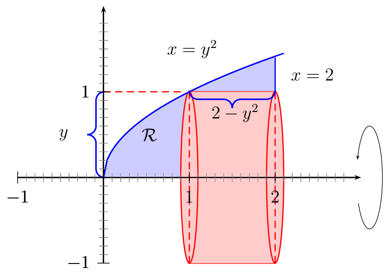

FIGURE 10

Solution: Since we will take a cross-section parallel to the $x-$axis, we will integrate with respect to y. Since x moves from $x = 0$ to $x = 2$, then y moves from $y = 0$ to $y = \sqrt{2}$. At each $y \in [0, \sqrt{2}]$, the cross-section of the region $\mathcal{R}$ parallel to $x-$axis generates a cylindrical surface of hight $2 - y^2$ and radius y. Since the area of the surface is $2\pi y(2 - y^2)$, the volume of the solid is

$$
\begin{aligned}
V &= \int_a^b 2\pi y f(y)\, dy \\
&= \int_0^{\sqrt{2}} 2\pi y(2 - y^2)\, dy \\
&= \int_0^{\sqrt{2}} 2\pi(2y - y^3)\, dy \\
&= 2\pi \left[2\frac{1}{2}y^2 - \frac{1}{4}y^4 \right]_0^{\sqrt{2}} \\
&= 2\pi \left[2 - \frac{1}{4}.4 \right] \\
&= 2\pi.
\end{aligned}
$$

Exercises 0.1. *In Exercises* $1-10$ *find the volume of the solid obtained by rotating the region bounded by the given curves about the given axis. Sketch the region, the solid, and a typical shell.*

(1) $y = x^2, \ y = 0, x = 1, \ x = 2,$ *about the* $y-$*axis*

(2) $y = x^2, \ y = 4, \ x = 0, \ x = 2,$ *about the* $y-$*axis*

(3) $x + y = 1, \ x = 0, \ y = 0,$ *about the* $x-$*axis*

(4) $x = y - y^2, \ x = 0,$ *about the* $x-$*axis*

(5) $y = 2x - x^2, \ y = 0, \ x = 0, \ x = 1,$ *about the* $y-$*axis*

(6) $y = x, \ y = 2 - x, \ x = 0,$ *about the* $x-$*axis*

(7) $y = x - 2, \ y = \sqrt{x-2},$ *about the* $y-$*axis*

(8) $y = \sqrt{x-1}, \ y = 0, \ x = 5,$ *about the* $y-$*axis*

(9) $y = \cos x x = 0, \ x = \frac{\pi}{4},$ *about the* $x-$*axis*

(10) $y = \frac{-1}{x}, \ y = 0, \ x = 1, \ y = 3,$ *about the* $x-$*axis*

