Absolute convergence

An important consideration when looking at the behavior of an arbitrary series

$$\sum_{n=1}^{\infty} a_n \qquad (1)$$

is the behavior of the related series

$$\sum_{n=1}^{\infty} |a_n| \qquad (2)$$

Of course, if all the terms of (1) are nonnegative, then (1) and (2) are the same series. In any case , (2) has all nonnegative terms, so we may use our results later to help determine whether or not it converges. Suppose that, by one method or another, we have shown that (2) converges. Then, since

$$0 \le a_n + |a_n| \le 2|a_n| \qquad (3)$$

for any n , we know, by the comparison test, that the series

$$\sum_{n=1}^{\infty} a_n + |a_n| \qquad (4)$$

converges. Hence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| \qquad (5)$$

converges since it is the difference of two convergent series. That is, the convergence of (2) implies the convergence of (1).

Theorem 1: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition 1: The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 1: The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

converges absolutely since the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges by p-series with $p = 2$. In particular, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges.

Leibniz's theorem: Suppose that $\sum_{n=1}^{\infty} a_n$ ia an alternating series for which $|a_{n+1}| \le |a_n|$ for $n = 1,2,3,\ldots$. If

$$\lim_{n\to\infty} |a_n| = 0,$$

then $\sum_{n=1}^{\infty} a_n$ converges.

Example 2: The alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

satisfies the conditions of Leibniz's theorem: If we let

$$a_n = \frac{(-1)^{n+1}}{n},$$

$n = 1, 2, 3, \dots$, then

$$|a_{n+1}| = \frac{1}{n+1} \le \frac{1}{n} = |a_n|$$

thus, as we claimed earlier, the alternating harmonic series converges.

Definition 2: A series which converges but does not converge absolutely is said ***converge conditionally***.

Example 3: The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots,$$

Known as alternating harmonic series, is not absolutely convergent since

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

is the harmonic series, which convergent. So the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is converge conditionally.

Problems:

1- 8. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

1. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$
2. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\sqrt{n}}$
3. $\sum_{n=1}^{\infty}\frac{3n^2-1}{4n^2+2}$
4. $\sum_{n=0}^{\infty}\frac{(-3)^{n}}{n!}$
5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+3}}$
6. $\sum_{n=3}^{\infty}(-1)^{n}\pi^{n}$
7. $\sum_{n=0}^{\infty}\frac{(-1)^{n}n!}{2^n}$
8. $\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^{n-1}$

9- 16. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

9. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2+1)}{3n^5-2}$

10. $\sum_{n=0}^{\infty} \frac{-3}{5^n}$

11. $\sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!}$

12. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$

13. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$

14. $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{3n-2}$

16. $\sum_{n=2}^{\infty} \frac{1-n}{2n^2}$