The Alternating Series Test

The Alternating Series Test applies only to a series with a very special form. Namely, the series must be of the form $\sum a_n$, where $a_n a_{n+1} < 0$. That is, the terms of the series alternate between positive and negative values. The test is relatively easy to apply and one can estimate the error between the actual sum of the series and a partial sum of the series.

The Alternating Series Test:

Let $\sum (-1)^n b_n$ be an alternating series with $b_n > 0$. The series converges if

1- $b_{n+1} \le b_n$ for all *n*, and

2- $\lim\limits_{n\to\infty} b_n = 0$

There are several observations one must keep in mind in applying the Alternating Series Test.

a. This is the first test we have that applies to a series where the terms are not eventually all positive. But one must identify the terms b_n so that they are positive.

b. One must verify Condition (a) when applying this test. While the test states the inequality holds for all n, it need only hold for all n sufficiently large because a finite number of terms does not change the convergence of any series. Sometimes it is easiest to define a function $f(x)$ such that $f(n) = b_n$ and show that *f* is decreasing for values of *x* sufficiently large.

c. The test can only show convergence. It can never be used to show divergence. However, if $\lim\limits_{n\to\infty} b_n \ne 0$, then the series does diverge, but because of the Divergence Test, not the Alternating Series Test.

d. In applying Condition (b), it is sometimes helpful to utilize the function used in applying Condition (a). One can use L'Hôpital's Rule on *f*.

e. In the event that the series converges, then the error between the actual sum *S* of the series and the partial sum s_n can be estimated by the inequality $|S - s_n| \le b_{n+1}$.

Example 1: Show that the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges and find the number of terms necessary to approximate the sum of the series correct to four decimal places.

Solution: (First determine b_n) Let $b_n = \frac{1}{\sqrt[3]{n}}$ for $n \ge 1$. (Check condition (a).) Since $n+1 \ge n$, it follows that

$$\sqrt[3]{n+1} \ge \sqrt[3]{n} \text{ and } b_{n+1} = \frac{1}{\sqrt[3]{n+1}} \le \frac{1}{\sqrt[3]{n}} = b_n .$$

(Check Condition (b).) Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0 .$$

So the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges by the Alternating Series Test.

(Determine the number of terms necessary to approximate the sum of the series correct to four decimal places.) Let S denote the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[3]{n}}$ and s_n denote a partial sum for the series. Then

$$\left|S-s_n\right|\le b_{n+1}=\frac{1}{\sqrt[3]{n+1}}\le 0.00001=\frac{1}{10^5}.$$

So $\sqrt[3]{n+1}\ge 10^5$ or $n\ge 10^{15}-1$.

Example 2: Determine whether or not the series $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{n^2+2}$ converges or diverges.

Solution: (First determine b_n.) Let $b_n=\frac{n+1}{n^2+2}>0$ for $n\ge 0$. (Check condition (a).) Let $f(x)=\frac{x+1}{x^2+2}$. Then

$$f'(x)=\frac{(x^2+2)-(x+1)2x}{(x^2+2)^2}=\frac{2-2x-x^2}{(x^2+2)^2}<0$$

for values of x sufficiently large. Therefore, if n is large enough, $b_{n+1}\le b_n$. (Check Condition (b).)

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x+1}{x^2+2}\overset{L.R.}{=}\lim_{x\to\infty}\frac{1}{2x}=0.$$

So

$$\lim_{n\to\infty}\frac{n+1}{n^2+2}=\lim_{n\to\infty} b_n=\lim_{n\to\infty} f(n)=\lim_{x\to\infty} f(x)=0.$$

Therefore the series $\sum_{n=0}^{\infty}(-1)^n\frac{n+1}{n^2+2}$ converges by the Alternating Series Test.

Example 3: Determine whether or not the series $\sum_{n=0}^{\infty}(-1)^n\frac{2n-3}{5n+1}$ converges or diverges.

Solution: (First determine b_n.) Let $b_n=\frac{2n-3}{5n+1}$ for $n\ge 0$. (Normally one would now check Condition (a), but looking ahead, check Condition (b) next.)

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{2n-3}{5n+1}=\lim_{n\to\infty}\frac{2-\frac{3}{n}}{5+\frac{1}{n}}=\frac{2}{5}\ne 0.$$

So

$$\lim_{n\to\infty}(-1)^n\frac{2n-3}{5n+1}$$

does not exist. By the Divergence Test the series $\sum_{n=0}^{\infty}(-1)^n\frac{2n-3}{5n+1}$ diverges.

Problems:

1-22. Determine whether the give series converges or diverges

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^{n} 2n}{n+3}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n+3}{n(n+1)}$
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