The Integral Test

The Integral Test is a method for determining the convergence or divergence of a series of positive terms. It has the advantage that, unlike some of the other tests we will discuss, it always provides an answer to the question of convergence. The drawbacks are:

1. There are a number of conditions that must be satisfied in order to apply the test;
2. One must be able to evaluate an improper integral, which may be difficult to accomplish for a given series.

The Integral Test:
Given a series $\sum a_n$ of positive terms define a function $f(x)$ such that $f(n)=a_n$ for all large values of n. If this function $f(x)$ is continuous, positive and decreasing for all $x \geq 1$, then the series $\sum a_n$ and the improper integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

The p-series: The p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

converges for $p>1$ and diverges for $p \leq 1$.

Example 1: Determine whether or not the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ converges or diverges.

Solution: (Notice that because $n>1$, $\ln n>0$ and so the series is a series of positive terms. This means that the Integral Test may apply provided one can identify an appropriate function f.)

Define $f(x)=\frac{1}{x\ln x}$. Then

$$f(n)=\frac{1}{n\ln n} \quad \text{for all} \quad n \geq 2.$$

For $x \geq 2$, $f(x)=\frac{1}{x\ln x}>0$ and

$$f'(x)=\frac{d}{dx}(x\ln x)^{-1}=-(x\ln x)^{-2}\frac{d}{dx}(x\ln x)=-(x\ln x)^{-2}(\ln x+1)<0.$$

So since f is differentiable, f is continuous on the interval $[2,\infty)$. Since the derivative of f is negative, f is decreasing on $[2,\infty)$. Let $u=\ln x$, then $x\,du=dx$

$$\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{t\to\infty}\int_2^{t}\frac{dx}{x\ln x}=\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u}=\lim_{t\to\infty}\ln u\Big|_{\ln 2}^{\ln t}=\lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)=\infty.$$

Since the improper integral diverges, the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges.

Example 2: Determine whether or not the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges or diverges.

Solution: Define $f(x)=\frac{1}{x^2+1}$. Then $f(n)=\frac{1}{n^2+1}$ for all $n \geq 1$. For $x \geq 1$, $f(x)=\frac{1}{x^2+1}>0$ and

$$f'(x) = \frac{d}{dx}\left(x^2+1\right)^{-1} = -\left(x^2+1\right)^{-2}\frac{d}{dx}\left(x^2+1\right) = -2x\left(x^2+1\right)^{-2} < 0 .$$

So since f is differentiable, f is continuous on the interval $\left[1,\infty\right)$. Since the derivative of f is negative, f is decreasing on the interval $\left[1,\infty\right)$.

$$\int_1^{\infty}\frac{dx}{x^2+1} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2+1} = \lim_{t\to\infty}\tan^{-1}x\Big|_1^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}1\right) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}.$$

Since the improper integral converges, the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges.

Example 3: The series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

diverges because it is a p-series with $p=\frac{1}{2}$.

Example 4: The series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges because it is a p-series with $p=2$.

Example 5: Determine whether or not the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges or diverges.

Solution: Define $f(x)=\frac{1}{x^2}$. Then $f(n)=\frac{1}{n^2}$ for $n\ge 1$. For $x\ge 1$, $f(x)>0$, $f'(x)>0$ and $f'(x)=\frac{-2}{x^2}$. Therefore, f is positive, decreasing and continuous for $x\ge 1$ meeting the conditions required for the integral test.

$$\int_1^{\infty}\frac{dx}{x^2} = \lim_{t\to\infty}\int_1^t\frac{dx}{x^2} = \lim_{t\to\infty}\left(\frac{-1}{x}\right)\Bigg|_1^t = \lim_{t\to\infty}\left(\frac{-1}{t}+1\right) = 1.$$

Since the improper integral converges, the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Problems:

1-24. Determine whether the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

3. $\sum_{n=1}^{\infty} \frac{8}{n^2}$

4. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

5. $\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$

6. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

7. $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$

8. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

9. $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

10. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

11. $\sum_{n=1}^{\infty} \frac{1}{(5+4n)}$

12. $\sum_{n=1}^{\infty} \frac{1}{1+3n^2}$

13. $\sum_{n=1}^{\infty} \frac{12}{(3+n)^{3/2}}$

14. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

15. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

16. $\sum_{n=1}^{\infty} n\, 3^{-n^2}$

17. $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$

18. $\sum_{n=4}^{\infty} \frac{n}{\ln n}$

19. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

20. $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{\pi}{n}$

21. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

22. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

23. $\sum_{n=1}^{\infty} \frac{\tan^{-1} x}{n^2+1}$

24. $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]}$

