The Radius of Convergence for a Power Series

Definition 1 : A ***power series*** is a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$, where a is a constant, $\{c_n\}_{n=0}^{\infty}$ is a sequence of numbers and x is a variable. Sometimes a power series is referred to as ***a power series centered about*** a (or ***a power series centered at*** a).

Example 1: Below are several examples of power series in which the various components of the series are identified. It is extremely important that are able to identify the components of a power series.

No.	**Series**	a	c_n
1	$\sum_{n=0}^{\infty} x^n$	$a=0$	for all $n \geq 0, c_n = 1$
2	$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$	$a=1$	for all $n \geq 0, c_n = \frac{1}{n!}$
3	$\sum_{n=0}^{\infty} n!(x+3)^n$	$a=-3$	for all $n \geq 0, c_n = n!$.
4	$\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n}$	$a=2$	$c_0 = 0$ and for $n \geq 1, c_n = \frac{(-1)^n}{n}$
5	$\sum_{n=2}^{\infty} n^2 (x+1)^n$	$a=-1$	$c_0 = 0, c_1 = 0$ and for all $n \geq 2, c_n = n^2$

Basic Question: For what values of x does the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converge?

Example 2: Consider the power series $\sum_{n=0}^{\infty} x^n$. For what values of x does this series converge?

Since x does not depend on n, this series is a Geometric Series with first term *1* and common ratio x. So the series converges for $|x|<1$, or $-1<x<1$. Furthermore,
if $|x|<1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Remarks: There are several of remarks that pertain to the basic question. These remarks are illustrated with the example $\sum_{n=0}^{\infty} x^n$:

a. Notice $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

b. The series $\sum_{n=0}^{\infty} c_n (x-a)^n$ always converges for $x=a$ since the terms of the series are all zero except for the first term. So for $x=a$, $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0$.

For $x=0$, $\sum_{n=0}^{\infty} x^n = 1+0+0^2+\cdots+0^n+\cdots = 1$.

c. For those values of *x* for which the power series converges, the power series sums to a number. So we can think of the power series as defining a function f whenever it converges. That is,

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for those values of *x* for which the power series converges.

For all $-1<x<1$, $f(x) \equiv \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

d. If x is a value for which the power series diverges, then the equality $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ in the previous observation does not hold.

Since $x=2>1$, the power series diverges for $x=2$. So $-1 = f(2) \neq \sum_{n=0}^{\infty} 2^n$. Since the power series does not converges (that is, it diverges) for $x=-1$, $\frac{1}{2} = f(-1) \neq \sum_{n=0}^{\infty} (-1)^n$.

Example 3: To begin answering the Basic Question, apply the Ratio Test:

1) Consider the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$. Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{n!(x-1)^{n+1}}{(n+1)n!(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1.$$

So the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ converges for all values of *x*. Further we may define $f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ for $-\infty < x < \infty$.

2) Consider the power series $\sum_{n=0}^{\infty} n!(x+3)^n$. Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+3)^{n+1}}{n!(x+3)^n}\right| = |x+3|\lim_{n\to\infty}(n+1) = \begin{cases} 0, & x=-3 \\ \infty, & x\neq -3 \end{cases}.$$

So the power series converges if and only if $x=-3$. The function $f(x) = \sum_{n=0}^{\infty} n!(x+3)^n$ is defined for only the single value $x=-3$.

3) Consider the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n}$. Applying the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x-2)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{n}{n+1} = |x-2|\,(1) = |x-2|.$$

So the power series converges for $|x-2|<1$ and diverges for $|x-2|>1$. No information is known for $|x-2|=1$. The function $f(x)=\sum_{n=1}^{\infty}(-1)^n\frac{(x-2)^n}{n}$ is defined for $|x-2|<1$ and undefined for $|x-2|>1$. It may or may not be defined for $|x-2|=1$.

4) Consider the power series $\sum_{n=2}^{\infty} n^2(x+1)^n$. Applying the Ration Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+1)^{n+1}}{n^2(x+1)^n}\right| = |x+1|\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = |x+1|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = |x+1|\,(1) = |x+1|.$$

So the power series converges for $|x+1|<1$ and diverges for $|x+1|>1$. No information is known for $|x+1|=1$. The function $f(x)=\sum_{n=2}^{\infty} n^2(x+1)^n$ is defined for $|x+1|<1$ and undefined for $|x+1|>1$. It may or may not be defined for $|x+1|=1$.

In general, the exactly one of the following cases happen for a given power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

- The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for all values of x.
- The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $x=a$ only.
- There is a number R for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

In the third case the number R is called the ***Radius of Convergence*** (sometimes abbreviated by ***ROC***) for the given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. In order to extend this notion to the other cases, define the radius of convergence in the first case to be ∞ and in the second case the radius of convergence is defined to be zero.

Remarks The ROC for each of the examples above is stated for completeness. Each is based on the work in the example.

No.	**Series**	**Radius of convergence of R**
1	$\sum_{n=0}^{\infty} x^n$	$R=1$
2	$\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$	$R=\infty$
3	$\sum_{n=0}^{\infty} n!(x+3)^n$	$R=0$

4	$\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{n}$	$R=1$
5	$\sum_{n=2}^{\infty} n^2 (x+1)^n$	$R=1$

Problems:

1-8. For each of the following power series, find the radius of convergence:

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$
3. $\sum_{n=1}^{\infty} n x^n$
4. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$
5. $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n+1}$
6. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$
7. $\sum_{n=1}^{\infty} n^3 x^{2n}$
8. $\sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2}$
9. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$
10. $\sum_{n=1}^{\infty} \frac{x^n}{n 5^n}$
11. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{(2n)!}$
12. $\sum_{n=1}^{\infty} 3^n x^n$
13. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$
14. $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n}$