The Ratio Test and Root Test

The Ratio Test provides a method to verify a given series converges absolutely. Unlike most of our previous tests, it applies to any series. Since absolute convergence implies convergence, the Ratio Test may be useful in showing a series converges. The drawback is that it sometimes fails to provide an answer.

The Ratio Test:

Let $\sum a_n$ be a given series. Then

1- If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ converges absolutely (and therefore converges);

2- If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ diverges;

3- If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, no information for $\sum a_n$ is provided by this test.

The Root Test:

Let $\sum a_n$ be a given series. Then

1- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely (and therefore converges);

2- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then $\sum a_n$ diverges;

3- If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$, no information for $\sum a_n$ is provided by this test.

Example 1: Determine whether the series $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 3^n}{n!}$ is absolutely convergent.

Solution:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}3^{n+1}}{(n+`1)!}}{\frac{(-1)^n 3^n}{n!}}\right| = \lim_{n\to\infty}\frac{3^{n+1}n!}{3^n(n+`1)!} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1.$$

So by the Ration Test the series $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 3^n}{n!}$ converges absolutely.

Example 2: Determine the whether the series

$$\sum_{n=0}^{\infty} a_n = 1 - \frac{3}{2} - \frac{3^2}{2^3} + \frac{3^3}{2^5} + \frac{3^4}{2^7} - \frac{3^5}{2^9} - \frac{3^6}{2^{11}} + \cdots$$

is convergent.

Solution:

1- Find a_n : $a_n = \frac{3^n}{2^{2n-1}}$ for $n \ge 1$.

2- Compute $\left|\frac{a_{n+1}}{a_n}\right|$: $\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \dfrac{\frac{3^{n+1}}{2^{2n+1}}}{\frac{3^n}{2^{2n-1}}} = \dfrac{3^{n+1}\cdot 2^{2n-1}}{2^{2n+1}3^n} = \dfrac{3}{4}.$

3- compute $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$: $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{3}{4}.$

4- The series converges since $L = \dfrac{3}{4} < 1$.

Example 3: Determine whether or not the series $\sum\limits_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$ converges.

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\left(\frac{3}{2}\right)^{n+1}}{n\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty}\frac{(n+1)\cdot\frac{3}{2}}{n}$$

$$= \frac{3}{2}\lim_{n\to\infty}\frac{n+1}{n} = \frac{3}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{3}{2} > 1.$$

So the series $\sum\limits_{n=1}^{\infty} n\left(\frac{3}{2}\right)^n$ diverges.

Example 4: Determine whether or not the series $\sum\limits_{n=1}^{\infty}\left(\frac{-n}{2n+1}\right)^n$ converges.

Solution:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-n}{2n+1}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1.$$

So by the Root Test, the given series converges absolutely and therefore it converges.

Example 5: Determine whether or not the series $\sum\limits_{n=1}^{\infty}\left(\frac{n^2}{2n+1}\right)^n$ converges.

Solution:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n^2}{2n+1}\right)^n\right|} = \lim_{n\to\infty}\left(\frac{n^2}{2n+1}\right) = \infty.$$

So the series $\sum\limits_{n=1}^{\infty}\left(\frac{n^2}{2n+1}\right)^n$ diverges.

Problem:

1-14. Use the Ratio test to conclude what you can about the convergence or divergence of $\sum\limits_{n=1}^{\infty} f(n)$ for each given $f(n)$:

1. $f(n)=\dfrac{3^n}{n^3}$

2. $f(n)=\dfrac{(-1)^n 3^{n-2}}{2^n}$

3. $f(n)=\dfrac{n}{(n+1)e^n}$

4. $f(n)=\dfrac{n\,7^n}{n!}$

5. $f(n)=\dfrac{10^{2n}}{(2n-1)!}$

6. $f(n)=\dfrac{(-1)^n 3^{n-1}}{n!}$

7. $f(n)=\dfrac{2^{3n}}{3^{2n}}$

8. $f(n)=n\left(\dfrac{2}{3}\right)^n$

9. $f(n)=\dfrac{(\sqrt{5}-1)^n}{n^2+1}$

10. $f(n)=\dfrac{n^2+7n+1}{n^2 n!}$

11. $f(n)=\dfrac{n^n}{3^n n!}$

12. $f(n)=\dfrac{5^n n!}{(2n)^n}$

13. $f(n)=\dfrac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$

14. $f(n)=\dfrac{n(n+2)}{(n+1)!}$

15-22. Use the Root test to conclude what you can about the convergence or divergence of $\sum_{n=1}^{\infty} f(n)$ for each given $f(n)$:

15. $f(n)=\dfrac{1}{n^n}$

16. $f(n)=\dfrac{2^n}{n^2}$

17. $f(n)=\left(1+\dfrac{1}{n}\right)^n$

18. $f(n)=\dfrac{(-1)^n}{[\ln(n+1)]^n}$

19. $f(n)=(-1)^{n+1}\left(\dfrac{n}{(3n+1)}\right)^n$

20. $f(n)=\dfrac{n^n}{(2n+1/n)^n}$

21. $f(n)=\dfrac{(-1)^n n^n}{(\ln n)^n},\quad n\geq 2$

22. $f(n)=\left(\dfrac{n}{n+1}\right)^{n^2}$