Representations the function as a power series

Further uses of the notion of 'changing the variable' are illustrated in the following examples. These examples show how to sum various series that are related to the Geometric Series. The technique applies to any series that can be summed, but at the moment we can only sum the Geometric Series. Specifically, recall

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}, \quad |t| < 1.$$

Example 1: Find a series that represents the function $f(x) = \frac{1}{1-x^2}$.

Solution. Letting $t = x^2$,

$$\frac{1}{1-x^2} = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n},$$

for $|t| < 1 \Leftrightarrow \left|x^2\right| < 1 \Leftrightarrow |x| < 1$.

Example 2: Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^{n+1}}$.

Solution. Setting $t = -\frac{x}{3}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty} t^n = \frac{1}{3}\left(\frac{1}{1-t}\right) = \frac{1}{3}\cdot\frac{1}{1-\left(-\frac{x}{3}\right)} = \frac{1}{3+x},$$

for $|t| < 1 \Leftrightarrow \left|-\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3$.

Example 3: Find a series that represents the function $f(x) = \frac{2x^2}{2-3x}$.

Solution. Setting $t = \frac{3x}{2} \Leftrightarrow x = \frac{2t}{3}$

$$f(x) = \frac{2x^2}{2-3x} = 2x^2\left(\frac{\frac{1}{2}}{1-\frac{3x}{2}}\right) = \left(\frac{2t}{3}\right)^2\left(\frac{1}{1-t}\right) = \left(\frac{4t^2}{9}\right)\left(\sum_{n=0}^{\infty} t^n\right)$$

$$= \frac{4}{9}\sum_{n=0}^{\infty} t^{n+2} = \frac{2^2}{3^2}\sum_{n=0}^{\infty}\left(\frac{3x}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^{n-2} x^n,$$

for $|t| < 1 \Leftrightarrow \left|\frac{3x}{2}\right| < 1 \Leftrightarrow |x| < \frac{2}{3}$.

Example 4: Find the sum of the series $1 + \sum_{n=1}^{\infty} 2x^n$.

Solution.

$$1 + \sum_{n=1}^{\infty} 2x^n = 1 + 2\sum_{n=1}^{\infty} x^n = 1 + 2\left(-1 + 1 + \sum_{n=1}^{\infty} x^n\right) = 1 - 2 + 2\sum_{n=0}^{\infty} x^n = -1 + \frac{2}{1-x} = \frac{-(1-x)+2}{1-x} = \frac{1+x}{1-x},$$

for $|x| < 1$.

Differentiating and Integrating Power Series

Since a power series can be considered as a function o its IOC, it is natural to ask: How does one perform certain operations on them that are customary on the function studied to date? These operations include differentiation and integration as well as certain arithmetic operations.

> ***Theorem*** *(Differentiating and Integrating power series Theorem)*
>
> Suppose $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$ (or possibly $R=\infty$), Then
>
> f is differentiable (and therefore continuous) on $(R-a, R+a)$ and
>
> a. $f'(x)=\sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$
>
> b. $\int f(x)dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}+K$, where K is the constant of integration for the indefinite integral.
>
> c. In both cases the radius of convergence is R, the same as the ROC of the original series.

Remarks There are several remarks that one needs to keep in mind when applying the above theorem.

- Differentiating a series is accomplished by differentiating term by term.
- Integrating a series is accomplished by integrating term by term.
- While the *radius* of convergence is the same for all the series, the *interval* of convergence may be different. That is to say, the endpoints of the interval of convergence for the original series may or may not be included in the IOC for the derivative or the IOC for the integral.

The following examples demonstrate how to apply the theorem in different circumstances.

Example 1: Find a power series representation for the function $f(x)=\tan^{-1} x$ and determine the interval of convergence.

Solution Recall that and we apply the Geometric series $\frac{1}{1-s}=\sum_{n=0}^{\infty} s^n$ and set $s=-t^2$

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \frac{dt}{1-\left(-t^2\right)} = \int_0^x \sum_{n=0}^{\infty}\left(-t^2\right)^n dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}dt$$

$$= \sum_{n=0}^{\infty}(-1)^n \left.\frac{t^{2n+1}}{2n+1}\right|_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

To determine the IOC for this power series, we know that the radius of convergence for the Geometric Series is $R=1$. So the ROC for the power series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$ is also $R=1$. Then the series converges for $|x|<1 \Leftrightarrow -1<x<1$. Now check the endpoints:

For $x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(-1)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ (Since $(-1)^{2n+1}=-1$ for all n)

This is an alternating series. Set $b_n=\frac{1}{2n+1}$. Then $b_n=\frac{1}{2n+1}>\frac{1}{2n+3}=b_{n+1}$, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n+1}=0.$$

So by the Alternating Series Test $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ converges

For $x=1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(1)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ which is also a convergent alternating series as shown above. So the interval of convergence is $[-1,1]$, or equivalently, $-1\le x\le 1$.

Note: Choosing $x=1$ yields an interesting expression for π as follows:

$$\frac{\pi}{4}=\tan^{-1}1=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{2n+1}+\cdots$$

This is an alternating series that can be used to approximate π.

Example 2: Find a power series representation for the function $f(x)=\frac{x}{(x+3)^2}$ and determine the radius of convergence for the power series.

Solution Note that $\frac{d}{dx}\left(\frac{1}{3+x}\right)=-\frac{1}{(3+x)^2}$ and

$$\frac{1}{3+x}=\frac{1}{3-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \text{for} \quad |x|<1.$$

So

$$\frac{x}{(x+3)^2}=-x\left[\frac{d}{dx}\left(\frac{1}{3+x}\right)\right]=-x\left[\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n\right]=-x\sum_{n=1}^{\infty}(-1)^n nx^{n-1}=\sum_{n=1}^{\infty}(-1)^{n+1} nx^n,$$

for $|x|<1$. So the ROC is $R=1$. (It turns out the interval of convergence is $-1<x<1$. Can you verify this?)

Example 3: Evaluate the indefinite integral $\int\frac{x\,dx}{1+x^5}$ as a power series.

Solution Consider

$$\frac{x}{1+x^5}=x\left(\frac{1}{1-(-x^5)}\right)=x\sum_{n=0}^{\infty}(-x^5)^n=\sum_{n=0}^{\infty}(-1)^n x^{5n+1}$$

for $|x|<1$. So

$$\int\frac{xdx}{1+x^5}=\int\sum_{n=0}^{\infty}(-1)^n x^{5n+1}dx=\sum_{n=0}^{\infty}(-1)^n\int x^{5n+1}dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{5n+2}}{5n+2}+K,$$

for some constant *K*.