Sequences

Definition 1*:** A ***sequence is an ordered list of numbers: $a, b, c, d, \ldots$. Each number is called a ***term of the sequence*** and is referred to by the number of its position in the listing. So in the listing above, a is the first term, b is the second term, c is the third term and so forth. If there are only a finite number of terms in the sequence, then the sequence is called a ***finite sequence.*** Otherwise there are an infinite number of terms in the sequence and the sequence is called an ***infinite sequence.***

***Notation*:** The following notation to indicate a sequence: $\{a_1, a_2, a_3, a_4, \ldots, a_n, \ldots\}$, or in a more compact form $\{a_n\}_{n=1}^{\infty}$.

In this second form, the role of n is that of a ***counter*** and indicates the order in which the terms are listed. It is called the ***index*** for the sequence. Notice that in this case we have "counted" the terms of the sequence corresponding to the positions they hold. That is to say, a_1 is the first term, a_2 is the second term, a_3 is the third term, and so forth. In general, a_n is the n^{th} term. However it is sometimes desirable to change the index. We could, for example, write the above sequence as:

$$\{a_{n+1}\}_{n=0}^{\infty}, \text{or} \quad \{a_{n-1}\}_{n=2}^{\infty}.$$

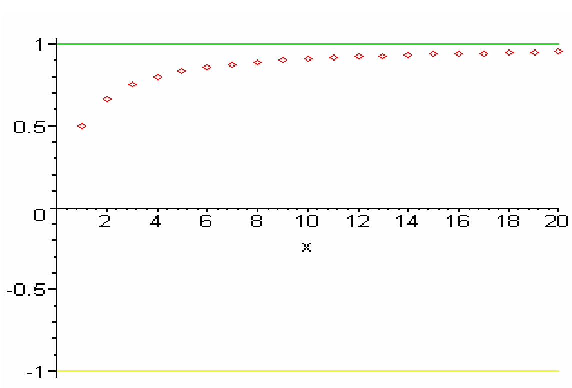
In both cases the terms of the sequence are the same and listed in the same order. So they are the same sequence. In fact, if we have two sequences, say $\{a_n\}_{n=1}^{\infty}$ and $\{b_i\}_{i=k}^{\infty}$, that are the same numbers listed in the same order, we say the sequences are equal and write

$$\{a_n\}_{n=1}^{\infty} = \{b_i\}_{i=k}^{\infty}.$$

Finally, there will be occasions in which the counter is unimportant to the discussion. In these cases we simply write $\{a_n\}$.

Example 1: The following examples of sequences written in various forms.

1) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$
2) $\left\{\dfrac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty} = \left\{\dfrac{-1}{2}, \dfrac{2}{3}, \dfrac{-3}{4}, \ldots, \dfrac{(-1)^n n}{n+1}, \ldots\right\}$
3) $\{n\}_{n=1}^{\infty} = \{1, 2, 3, \ldots, n, \ldots\}$.



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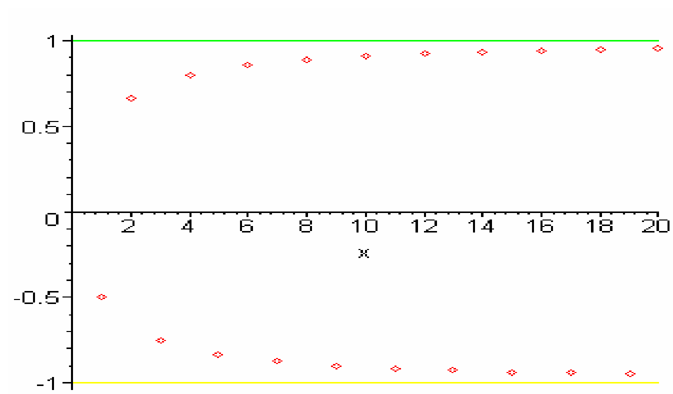

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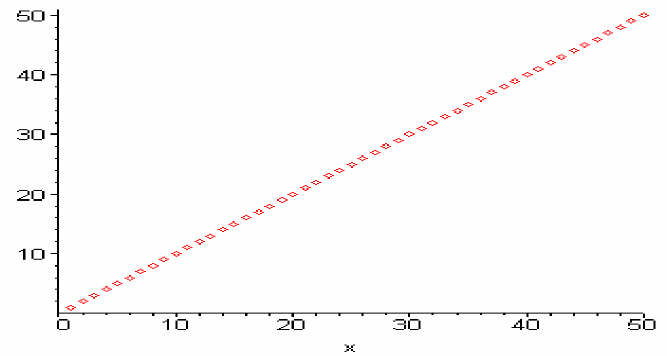

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There are several examples of a sequence can be defined. Each has its place in mathematics.

- *Defining a sequence by a statement:*
  1) The terms of the sequence are the squares of the number of the term. This statement yields the sequence:

$$\{n^3\}_{n=1}^{\infty} = \{1, 8, 27, \ldots, n^3, \ldots\}$$

  2) The n^{th} term of the sequence is the n^{th} term in the decimal expansion of the number π . This statement yields the sequence.

$$\{p_n\}_{n=1}^{\infty} = \{3, 1, 4, 1, 5, 9, \ldots\}$$

- *Defining a sequence by an equation:*
  1) The sequence $\{f_n\}_{n=1}^{\infty}$ is defined by

$$f_1 = 1; f_2 = 1; f_n = f_{n-1} + f_{n-2}, n \geq 3 .$$

  2) *This equation yields the sequence*

$$\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$$

This sequence is known as the Fibonacci sequence.

- *Defining a sequence by a formula:*

Suppose $y = f(x)$ is a function. Then we can define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_n = f(n)$ (provided the quantity $f(n)$ is defined). In Example 1 (1), choose $f(x) = \dfrac{x}{x+1}$. For Example 1 (3), there are several functions that can be used depending on which form you view the sequence. For $\{n\}_{n=1}^{\infty}$, choose $f(x) = x$.

Definition 2: Let $\{a_n\}$ denote a sequence. If there is a unique finite number L for which the terms of the sequence a_n are arbitrarily close to L for all n sufficiently large, then $\{a_n\}$ is said to ***converge*** to L and we write $\lim_{n\to\infty} a_n = L$.

The number L is called the ***limit of the sequence*** $\{a_n\}$. If the number L is not known or is unimportant to the discussion, then we say simply $\{a_n\}$ ***converges***, or $\{a_n\}$ is ***convergent***. If no such number L exists, then we say $\{a_n\}$ is ***divergent***, or $\{a_n\}$ ***diverges***. So every sequence either converges or diverges, but no sequence does both.

Example 2: Consider

$$\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

Here $a_n = \dfrac{1}{n^2}$ for each n. Clearly, if n is large, $a_n = \dfrac{1}{n^2}$ is arbitrarily close to zero. So we write $\lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$ and we say $\left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$ converges to zero.

Example 3: Consider $\left\{\sqrt{n}\right\}_{n=1}^{\infty} = \left\{1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$. Here $a_n = \sqrt{n}$ for each n. As the values of n get large, $a_n = \sqrt{n}$ become arbitrarily large also and therefore the terms of the sequence do not get close to any finite number. So we say $\left\{\sqrt{n}\right\}_{n=1}^{\infty}$ diverges.

Example 4: Consider

$$\left\{1 + \frac{(-1)^n}{2^n}\right\}_{n=0}^{\infty} = \left\{1, \frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \ldots, 1 + \left(-\frac{1}{2}\right)^n, \ldots\right\}.$$

Here $a_n = 1 + \dfrac{(-1)^n}{2^n} = 1 + (\dfrac{1}{2})^n$ for each n. For large values of n, $(-1/2)^n$ is numerically close to zero (even though the signs of these numbers alternate between positive and negative). So for large n, a_n is arbitrarily close to one. So we write $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (1 + \dfrac{(-1)^n}{2^n}) = 1$ and we say $\left\{1 + \dfrac{(-1)^n}{2^n}\right\}_{n=0}^{\infty}$ converges to one.

Example 5: Consider $\{(-1)^n\}_{n=2}^{\infty} = \{1, -1, 1, -1, \ldots, (-1)^n, \ldots\}$.

Here $a_n = (-1)^n$ for each n. As n becomes large, $a_n = (-1)^n$ simply alternates between l and $-l$. Therefore the terms of the sequence do not get close to any unique (one) number. Therefore we say $\{(-1)^n\}_{n=2}^{\infty}$ diverges.

Methods of Determining Convergence or Divergence of a Sequence

1- **Using limits of functions:**

Theorem 1: (Modeling Theorem). Let $\{a_n\}$ be a sequence, let $f(n) = a_n$, and suppose that $f(x)$ exists for every real number $x \geq 1$.

(i) $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} f(n) = L$.

(ii) $\lim\limits_{x\to\infty} f(x) = \infty$ (or $-\infty$), then $\lim\limits_{n\to\infty} f(n) = \infty$ (or $-\infty$).

Example 6: Consider $\left\{\dfrac{n^2}{n^2+1}\right\}_{n=1}^{\infty}$. Define $f(x) = \dfrac{x^2}{x^2+1}$ for $x > 0$. Using L'Hôpital's Rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2+1} \overset{L.R.}{=} \lim_{x\to\infty} \frac{2x}{2x} \overset{L.R.}{=} \lim_{x\to\infty} \frac{2}{2} = 1.$$

So $\left\{\dfrac{n^2}{n^2+1}\right\}_{n=1}^{\infty}$ converges to 1.

Example 7: Consider $\{\sqrt{n}\}_{n=3}^{\infty}$. Define $f(x) = \sqrt{x}$ for $x \geq 3$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{x} = \infty .$$

So $\{\sqrt{n}\}_{n=3}^{\infty}$ diverges.

2- Limit Laws for Sequences:

Just as there are rules for evaluating limits of functions, there are similar rules for evaluating limits of sequences. They are listed below.

Theorem 2: Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant. Then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} c = c$; $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$
5. $\lim_{n\to\infty}(a_n)^p = [\lim_{n\to\infty} a_n]^p$
6. If $a_n \leq b_n$ for all but a finite number of terms, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Theorem 3: Let r be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r \leq -1 \end{cases}.$$

DNE means does not exists .

Example 8: Consider $\left\{\frac{2n}{n^2+4}\right\}_{n=1}^{\infty}$. (Using Number 4, 3, 1, and 5)

$$\lim_{n\to\infty} \frac{2n}{n^2+4} = \lim_{n\to\infty} \frac{2n}{n^2+4} \cdot \frac{n^{-2}}{n^{-2}} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1+\frac{4}{n^2}} = \frac{\lim_{n\to\infty} \frac{2}{n}}{\lim_{n\to\infty}\left(1+\frac{4}{n^2}\right)} = \frac{2\lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1 + 4\lim_{n\to\infty} \frac{1}{n^2}}$$

$$= \frac{2\lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 1 + 4\left(\lim_{n\to\infty} \frac{1}{n}\right)^2} = \frac{2\cdot 0}{1+4(0)^2} = 0.$$

3- Comparing with other sequences:

Theorem 4: *(*The Squeeze Theorem) If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences and $a_n \leq b_n \leq c_n$ for every n and if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Example 9: Consider $\{n!/n^n\}_{n=1}^{\infty}$. (Recall $n! = 1\cdot 2\cdot 3 \cdots n$). Observe

$$0 \leq \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots(n-1)n}{\underbrace{n\cdot n \cdots n}_{n-times}} = \frac{1}{n}\frac{2}{n}\frac{3}{n}\cdots\frac{n-1}{n}\frac{n}{n} \leq \frac{1}{n}.$$

The sequence $\{0\}_{n=1}^{\infty}$ is a constant sequence and converges to zero. The sequence $\{1/n\}_{n=1}^{\infty}$ also converges to zero. (See Example 1 above.) So by the Squeeze Theorem, $\{n!/n^n\}_{n=1}^{\infty}$ converges to zero.

Example 10: Consider $\{(-1)^n/2^n\}_{n=0}^{\infty}$.

Since $\lim_{n\to\infty}\frac{1}{2^n}=0$, the sequence $\{1/2^n\}_{n=0}^{\infty}$ converges to zero. So the sequence $\{(-1)^n/2^n\}_{n=0}^{\infty}$ also converges to zero via the Squeeze Theorem.

Theorem 5: Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty}|a_n|=0$, then $\lim_{n\to\infty}a_n=0$.

Example 11: Consider $\left\{\frac{n\cos n}{n^2+1}\right\}_{n=1}^{\infty}$. We know that $0\le\left|\frac{n\cos n}{n^2+1}\right|\le\frac{n}{n^2+1}$. Furthermore, the sequence $\{0\}_{n=1}^{\infty}$ converges to zero because $\lim_{n\to\infty}0=0$. Since

$$\lim_{n\to\infty}\frac{n}{n^2+1}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}}=\frac{0}{1+0}=0,$$

the sequence $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ also converges to zero. Therefore, $\left\{\frac{n\cos n}{n^2+1}\right\}_{n=1}^{\infty}$ converges to zero via the Squeeze Theorem.

4- **Bounded Monotone Sequences:**

Definition 3: Let $\{a_n\}$ be a sequence. If $a_n\le a_{n+1}$ for all $n\ge1$ but a finite number of terms, then the sequence $\{a_n\}$ is said to be (monotone) ***increasing***. If $a_n\ge a_{n+1}$ for all $n\ge1$, but a finite number of terms, the sequence $\{a_n\}$ is said to be (monotone) ***decreasing***.
Finally, the sequence $\{a_n\}$ is said to be ***monotone*** provided it is either increasing or decreasing.
The sequence $\{a_n\}$ is said to be ***bounded above*** (or ***bounded below***) provided there is a number M for which $a_n\le M$ (or $a_n\ge M$) for all values of n. The sequence $\{a_n\}$ is said to be ***bounded*** provided it is both bounded above and bounded below.

The relevance to convergence is given by the following theorem

Theorem 6: A bounded, monotone sequence converges.

Example 12: Show that the sequence $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$ is decreasing.

Solution 1. We must show that $a_n\le a_{n+1}$ $n\ge1$ that is,

$$\frac{1}{n+1}>\frac{1}{n+2}$$

for all $n\ge1$.

Solution 2. Consider the function $f(x)=\frac{1}{x+1}$

$$f'(x) = \frac{-1}{(x+1)^2} < 0 \qquad \text{whenever } x \geq 1$$

Thus, f is decreasing on $(0,\infty)$ and so $f(n) > f(n+1)$. Therefore, $\{a_n\}$ is decreasing.

Problems:

1. Consider the sequence $1,3,5,7,\mathbf{K}$

a) If $\{a_n\}_{n=1}^{\infty}$ denotes this sequence, then

$$a_1 = -----,\ a_4 = -----,\ a_7 = -----$$

The general term is $a_n = -----$.

a) If $\{b_n\}_{n=1}^{\infty}$ denotes this sequence, then

$$b_0 = -----,\ b_4 = -----,\ b_8 = -----$$

The general term is $b_n = -----$.

2-10. Find the limit of the given sequence it is exist

2. $\left\{\dfrac{n}{n+2}\right\}$

3. $\left\{\dfrac{\ln n}{n+2}\right\}$

4. $\left\{\dfrac{(-1)^n}{n+1}\right\}$

5. $\left\{\dfrac{\sin n}{n+1}\right\}$

6. $\left\{\left(\dfrac{1}{2}\right)^n\right\}$

7. $\left\{1+\left(-\dfrac{1}{2}\right)^n\right\}$

8. $\left\{\dfrac{n}{e^n}\right\}$

9. $\left\{\sqrt[n]{n}\right\}$

10. $\left\{(-1)^n\right\}$

11. $\left\{\dfrac{n!}{n^n}\right\}$

12. $\left\{\dfrac{n!}{(n+1)!}\right\}_{n=0}^{\infty}$

13. $\left\{\dfrac{\ln n}{n}\right\}_{n=2}^{\infty}$

14. $\left\{\ln\left(\dfrac{1}{n}\right)\right\}_{n=1}^{\infty}$

15. $\left\{1+(-1)^n\right\}$

16. $\left\{\dfrac{\ln n}{n}\right\}_{n=1}^{\infty}$

17. $\left\{n\sin\dfrac{p}{n}\right\}_{n=1}^{\infty}$

18-21. Find the general term of the sequence, staring with $n=1$, determine the sequence converges, and if so find its limit.

18. $\dfrac{1}{2},\dfrac{3}{4},\dfrac{5}{6},\dfrac{7}{8},\mathbf{K}$

19. $(1-\dfrac{1}{2}),(\dfrac{1}{3}-\dfrac{1}{2}),(\dfrac{1}{3}-\dfrac{1}{4}),(\dfrac{1}{5}-\dfrac{1}{4}),\mathbf{K}$

20. $\dfrac{1}{2},\dfrac{-1}{4},\dfrac{1}{8},\dfrac{-1}{16},\mathbf{K}$

21. $3,\dfrac{3}{2},\dfrac{3}{2^2},\dfrac{3}{2^3},\mathbf{K}$

22-27. Determine whether the given sequence is increasing, decreasing, or not monotonic.

22. $\left\{\dfrac{n}{n+1}\right\}$

23. $\left\{\dfrac{e^n}{(n+1)!}\right\}$

24. $\left\{\dfrac{1}{n!}\right\}$

25. $\left\{\dfrac{1}{3^n}\right\}$

26. $\left\{\cos nx\right\}$

27. $\left\{(-2)^n\right\}$

28-29. Determine the general term of the sequences:

28. $\dfrac{1}{2},\dfrac{1}{12},\dfrac{1}{30},\dfrac{1}{56},\dfrac{1}{90},\ldots$

29. $\dfrac{1}{5^3},\dfrac{3}{5^5},\dfrac{5}{5^7},\dfrac{7}{5^9},\dfrac{9}{5^{11}},\ldots$