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Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions

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Abstract

We develop a generalized quasilinearization technique to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the forced Duffing equation with discontinuous type integral boundary conditions.

Keywords: Duffing equation; Integral boundary conditions; Quasilinearization; Quadratic convergence

1. Introduction

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, see for example [14,15,24]. In fact, boundary value problems involving integral boundary conditions have received considerable attention, see for instance, [7,9–12, 16,17,25]. For more details of the integral boundary conditions and important applications of the Duffing equation, we refer the reader to a recent reference [2] and the references therein.

In this paper, we continue the study of the forced Duffing equation with integral boundary conditions initiated in [2] by addressing a boundary value problem with discontinuous type integral boundary conditions. This consideration corresponds to a situation when some nonlinear forcing term is present at an intermediate point of the boundary segment and thereby generates a discontinuity in the integral boundary conditions. The results obtained under a new concept of discontinuous type integral boundary conditions in the present configuration are new and improve the earlier results. Precisely, we consider the following boundary value problem

$$\begin{cases} u''(t) + \sigma u'(t) + f(t, u(t)) = 0, \quad t \in [0,1], \sigma \in \mathbb{R} - \{0\}, \\ u(0) - \mu_1 u'(0) = g_1(u(1/2)) + \int_0^{\frac{1}{2}-} q_1(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(u(s))\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = g_2(u(1/2)) + \int_0^{\frac{1}{2}-} q_2(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_2(u(s))\mathrm{d}s, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, q_i are continuous functions on $(0, 1/2)$ and $(1/2, 1)$ and μ_i are nonnegative constants $(i = 1, 2)$.

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The method of quasilinearization (QSL) is applied to obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem (1.1). The QSL technique provides an elegant and easier approach for obtaining the quadratic convergence of the sequences of iterates. For the details of this method, see [1,3–6,8,13,18–23] and the references therein.

2. Preliminaries

Definition 2.1. A function $\alpha \in C^2[0,1]$ is a lower solution of (1.1) if

$$\alpha''(t) + \sigma\alpha'(t) + f(t,\alpha(t)) \geq 0, \quad t \in [0,1],$$
$$\alpha(0) - \mu_1\alpha'(0) \leq g_1(\alpha(1/2)) + \int_0^{\frac{1}{2}-} q_1(\alpha(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(\alpha(s))\mathrm{d}s,$$
$$\alpha(1) + \mu_2\alpha'(1) \leq g_2(\alpha(1/2)) + \int_0^{\frac{1}{2}-} q_2(\alpha(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_2(\alpha(s))\mathrm{d}s.$$

Similarly, $\beta \in C^2[0,1]$ is an upper solution of (1.1) if the inequalities in the definition of lower solution are reversed.

Since the associated homogeneous problem of (1.1) has only the trivial solution, therefore, by Green's function methods, the solution $u(t)$ of (1.1) can be written as

$$\begin{aligned} u(t) = {} & \frac{1}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}} \\ & \times \Bigg[((-1+\sigma\mu_2)\mathrm{e}^{-\sigma} + \mathrm{e}^{-\sigma t}) \left\{ g_1(u(1/2)) + \int_0^{\frac{1}{2}-} q_1(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(u(s))\mathrm{d}s \right\} \\ & + ((1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}) \left\{ g_2(u(1/2)) + \int_0^{\frac{1}{2}-} q_2(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_2(u(s))\mathrm{d}s \right\} \Bigg] \\ & + \int_0^1 G(t,s) f(s,u(s))\mathrm{d}s, \end{aligned}$$

where

$$G(t,s) = \Lambda \begin{cases} [(1-\sigma\mu_2) - \mathrm{e}^{\sigma(1-s)}][(1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}], & 0 \leq t \leq s, \\ [(1-\sigma\mu_2) - \mathrm{e}^{\sigma(1-t)}][(1+\sigma\mu_1) - \mathrm{e}^{-\sigma s}], & s \leq t \leq 1, \end{cases}$$
$$\Lambda = \frac{\mathrm{e}^{\sigma s}}{\sigma[(1-\sigma\mu_2) - (1+\sigma\mu_1)\mathrm{e}^{\sigma}]}.$$

We note that $G(t,s) > 0$ on $(0,1) \times (0,1)$. Let us prove some basic results which are needed to establish the main result.

Theorem 2.1. *Let α and β be lower and upper solutions of the boundary value problem (1.1) respectively. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be such that $f_u(t,u) < 0$ and q_i are continuous functions on $(0,1/2)$ and $(1/2,1)$ satisfying one sided Lipschitz conditions: $q_i(u) - q_i(v) \leq L_i(u-v)$, $0 \leq L_i < 1$, and $g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying one sided Lipschitz conditions: $g_i(u) - g_i(v) \leq L_i^*(u-v)$, $0 \leq L_i^* < 1$, $i = 1,2$. Then $\alpha(t) \leq \beta(t)$.*

Proof. Set $x(t) = \alpha(t) - \beta(t)$, $t \in [0,1]$ so that

$$\begin{cases} x(0) - \mu_1 x'(0) \leq g_1(\alpha(1/2)) - g_1(\beta(1/2)) + \displaystyle\int_0^{\frac{1}{2}-} [q_1(\alpha(s)) - q_1(\beta(s))]\mathrm{d}s \\ \qquad + \displaystyle\int_{\frac{1}{2}+}^1 [q_1(\alpha(s)) - q_1(\beta(s))]\mathrm{d}s, \\ x(1) + \mu_2 x'(1) \leq g_2(\alpha(1/2)) - g_2(\beta(1/2)) + \displaystyle\int_0^{\frac{1}{2}-} [q_2(\alpha(s)) - q_2(\beta(s))]\mathrm{d}s \\ \qquad + \displaystyle\int_{\frac{1}{2}+}^1 [q_2(\alpha(s)) - q_2(\beta(s))]\mathrm{d}s. \end{cases} \tag{2.1}$$

For the sake of contradiction, suppose that $x(t) > 0$ for $t \in [0,1]$. Then $x(t)$ has a positive maximum at some $t_0 \in [0,1]$. If $t_0 \in (0,1)$, then $x(t_0) > 0$, $x'(t_0) = 0$ and $x''(t_0) \le 0$. In view of the decreasing property of the function $f(t,u)$ in u, it follows that

$$x''(t_0) + \sigma x'(t_0) = \alpha''(t_0) + \sigma\alpha'(t_0) - (\beta''(t_0) + \sigma\beta'(t_0)) \ge -f(t_0,\alpha(t_0)) + f(t_0,\beta(t_0)) > 0,$$

which is a contradiction. If $t_0 = 0$, then $x(0) > 0$, $x'(0) = 0$. Using (2.1) together with the assumption that q_1 and g_1 satisfy one sided Lipschitz conditions, we obtain the contradiction

$$\begin{aligned} x(0) &= x(0) - \mu_1 x'(0) \le g_1(\alpha(1/2)) - g_1(\beta(1/2)) + \int_0^{\frac{1}{2}-} [q_1(\alpha(s)) - q_1(\beta(s))]\mathrm{d}s \\ &\quad + \int_{\frac{1}{2}+}^1 [q_1(\alpha(s)) - q_1(\beta(s))]\mathrm{d}s \\ &\le L_1^* x(1/2) + \frac{1}{2}L_1 \max_{t\in(0,1/2)} x(t) + \frac{1}{2}L_1 \max_{t\in(1/2,1)} x(t) = L_1^* x(1/2) + L_1 x(0) \\ &< x(0). \end{aligned}$$

A similar contradiction occurs at $t_0 = 1$. Hence $\alpha(t) \le \beta(t)$, $t \in [0,1]$. □

Theorem 2.2. *Assume that α and β are lower and upper solutions of the boundary value problem (1.1) respectively such that $\alpha(t) \le \beta(t)$. If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and q_i are continuous functions on $(0,1/2)$ and $(1/2,1)$ with g_i and q_i satisfying one sided Lipschitz conditions, then there exists a solution $u(t)$ of (1.1) such that $\alpha(t) \le u(t) \le \beta(t)$, $t \in [0,1]$.*

Proof. Let us define $F(t,u)$, $\hat{g}_i(u)$ and $Q_i(u)$ by

$$F(t,u) = \begin{cases} f(t,\beta(t)) - \dfrac{u-\beta(t)}{1+|u-\beta|}, & \text{if } u > \beta, \\ f(t,u(t)), & \text{if } \alpha \le u \le \beta, \\ f(t,\alpha(t)) - \dfrac{u-\alpha(t)}{1+|u-\alpha|}, & \text{if } u < \alpha, \end{cases}$$

$$\hat{g}_i(u) = \begin{cases} g_i(\beta(1/2)), & \text{if } u > \beta(1/2), \\ g_i(u(t)), & \text{if } \alpha(1/2) \le u \le \beta(1/2), \\ g_i(\alpha(1/2)), & \text{if } u < \alpha(1/2), \end{cases}$$

and

$$Q_i(u) = \begin{cases} q_i(\beta(t)), & \text{if } u > \beta, \\ q_i(u(t)), & \text{if } \alpha \le u \le \beta, \\ q_i(\alpha(t)), & \text{if } u < \alpha. \end{cases}$$

Since $F(t,u)$, $\hat{g}_i(u)$ and $Q_i(u)$ are continuous and bounded, it follows that there exists a solution $u(t)$ of the problem

$$\begin{cases} u''(t) + \sigma u'(t) + F(t,u) = 0, \quad t \in [0,1], \\ u(0) - \mu_1 u'(0) = \hat{g}_1(u(1/2)) + \displaystyle\int_0^{\frac{1}{2}-} Q_1(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(u(s))\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \hat{g}_2(u(1/2)) + \displaystyle\int_0^{\frac{1}{2}-} Q_2(u(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(u(s))\mathrm{d}s. \end{cases} \tag{2.2}$$

In relation to (2.2), we have

$$\alpha''(t) + \sigma\alpha'(t) + F(t,\alpha(t)) = \alpha''(t) + \sigma\alpha'(t) + f(t,\alpha(t)) \ge 0, \quad t \in [0,1],$$

$$\begin{aligned} \alpha(0) - \mu_1\alpha'(0) &\le g_1(\alpha(1/2)) + \int_0^{\frac{1}{2}-} q_1(\alpha(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(\alpha(s))\mathrm{d}s \\ &= \hat{g}_1(\alpha(1/2)) + \int_0^{\frac{1}{2}-} Q_1(\alpha(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(\alpha(s))\mathrm{d}s, \end{aligned}$$

$$\begin{aligned}\alpha(1)+\mu_2\alpha'(1) &\leq g_2(\alpha(1/2))+\int_0^{\frac{1}{2}-} q_2(\alpha(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 q_2(\alpha(s))\mathrm{d}s\\ &= \hat{g}_2(\alpha(1/2))+\int_0^{\frac{1}{2}-} Q_2(\alpha(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 Q_2(\alpha(s))\mathrm{d}s\end{aligned}$$

and

$$\beta''(t)+\sigma\beta'(t)+F(t,\beta(t))=\beta''(t)+\sigma\beta'(t)+f(t,\beta(t))\leq 0,\quad t\in[0,1],$$

$$\begin{aligned}\beta(0)-\mu_1\beta'(0) &\geq g_1(\beta(1/2))+\int_0^{\frac{1}{2}-} q_1(\beta(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 q_1(\beta(s))\mathrm{d}s\\ &= \hat{g}_1(\beta(1/2))+\int_0^{\frac{1}{2}-} Q_1(\beta(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 Q_1(\beta(s))\mathrm{d}s,\\ \beta(1)+\mu_2\beta'(1) &\geq g_2(\beta(1/2))+\int_0^{\frac{1}{2}-} q_2(\beta(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 q_2(\beta(s))\mathrm{d}s\\ &= \hat{g}_2(\beta(1/2))+\int_0^{\frac{1}{2}-} Q_2(\beta(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 Q_2(\beta(s))\mathrm{d}s,\end{aligned}$$

which imply that α and β are lower and upper solutions of (2.2) respectively. By definition of $F(t,u)$, it follows that any solution $u\in[\alpha,\beta]$ of (2.2) is indeed a solution of (1.1). Thus, we just need to show that any solution $u(t)$ of (2.2) satisfies $\alpha(t)\leq u(t)\leq\beta(t)$, $t\in[0,1]$. Let us assume that $\alpha(t)>u(t)$ on $[0,1]$. Then the function $y(t)=\alpha(t)-u(t)$ has a positive maximum at some $t=t_0\in[0,1]$. If $t_0\in(0,1)$, then $y(t_0)>0$, $y'(t_0)=0$, $y''(t_0)\leq 0$. On the other hand,

$$\begin{aligned}y''(t_0)+\sigma y'(t_0) &= \alpha''(t_0)+\sigma\alpha'(t_0)-[u''(t_0)+\sigma u'(t_0)]\\ &\geq -F(t_0,\alpha(t_0))+F(t_0,u(t_0))\\ &= -f(t_0,\alpha(t_0))+f(t_0,\alpha(t_0))-\frac{u(t_0)-\alpha(t_0)}{1+|u(t_0)-\alpha(t_0)|}>0,\end{aligned}$$

which contradicts our assumption. If $t_0=0$, then $y(0)>0$, $y'(0)=0$ and

$$\begin{aligned}y(0)=\alpha(0)-u(0) &\leq \mu_1 y'(0)+g_1(\alpha(1/2))-\hat{g}_1(u(1/2))\\ &\quad+\int_0^{\frac{1}{2}-}[q_1(\alpha(s))-Q_1(u(s))]\mathrm{d}s+\int_{\frac{1}{2}+}^1[q_1(\alpha(s))-Q_1(u(s))]\mathrm{d}s\\ &= g_1(\alpha(1/2))-\hat{g}_1(u(1/2))+\int_0^{\frac{1}{2}-}[q_1(\alpha(s))-Q_1(u(s))]\mathrm{d}s\\ &\quad+\int_{\frac{1}{2}+}^1[q_1(\alpha(s))-Q_1(u(s))]\mathrm{d}s.\end{aligned}\tag{2.3}$$

If $u(t)<\alpha(t)$, then $\hat{g}_1(u(1/2))=g_1(\alpha(1/2))$ and $Q_1(u(s))=q_1(\alpha(s))$ and consequently (2.3) yields a contradiction $y(0)\leq 0$. If $u(t)>\beta(t)$, then $\hat{g}_1(u(1/2))=g_1(\beta(1/2))$ and $Q_1(u(s))=q_1(\beta(s))$. Since g_1 and q_1 satisfy one sided Lipschitz conditions, we have $g_1(\alpha(1/2))-\hat{g}_1(u(1/2))=(g_1(\alpha(1/2))-g_1(\beta(1/2)))\leq L_1^*(\alpha(1/2)-\beta(1/2))\leq 0$ and $q_1(\alpha(s))-Q_1(u(s))=(q_1(\alpha(s))-q_1(\beta(s)))\leq L_1(\alpha(s)-\beta(s))$, which, on substituting in (2.3), yields

$$\begin{aligned}y(0) &\leq L_1\int_0^{\frac{1}{2}-}(\alpha(s)-\beta(s))\mathrm{d}s+L_1\int_{\frac{1}{2}+}^1(\alpha(s)-\beta(s))\mathrm{d}s\\ &\leq \frac{L_1}{2}\max_{t\in(0,1/2)}(\alpha(t)-\beta(t))+\frac{L_1}{2}\max_{t\in(1/2,1)}(\alpha(t)-\beta(t))=L_1(\alpha(0)-\beta(0))\leq 0,\end{aligned}$$

which is a contradiction. If $\alpha(t)\leq u(t)\leq\beta(t)$, then $\hat{g}_1(u(1/2))=g_1(u(1/2))$, $Q_1(u(s))=q_1(u(s))$ and $g_1(\alpha(1/2))-\hat{g}_1(u(1/2))=g_1(\alpha(1/2))-g_1(u(1/2))\leq L_1^*(\alpha(1/2)-u(1/2))\leq L_1^*(\alpha(1/2)-\alpha(1/2))=0$,

$q_1(\alpha(s)) - Q_1(u(s)) = q_1(\alpha(s)) - q_1(u(s)) \le L_1(\alpha(s) - u(s)) \le L_1(\alpha(s) - \alpha(s)) = 0$ which together with (2.3) leads to a contradiction $y(0) \le 0$. In a similar manner, we have a contradiction at $t_0 = 1$. Thus, $\alpha(t) \le u(t), t \in [0, 1]$. Analogously, it can be shown that $u(t) \le \beta(t), t \in [0, 1]$. Hence we conclude that $\alpha(t) \le u(t) \le \beta(t)$, $t \in [0, 1]$. □

3. Main result

Theorem 3.1. *Assume that*

(A_1) α *and* $\beta \in C^2[0, 1]$ *are respectively lower and upper solutions of* (1.1) *such that* $\alpha(t) \le \beta(t)$;
(A_2) $f(t, u) \in C^2([0, 1] \times \mathbb{R})$ *be such that* $f_u(t, u) < 0$ *and* $(f_{uu}(t, u) + \phi_{uu}(t, u)) \ge 0$, *where* $\phi_{uu}(t, u) \ge 0$ *for some continuous function* $\phi(t, u)$ *on* $[0, 1] \times \mathbb{R}$;
(A_3) q_i *are continuous functions on* $(0, 1/2)$ *and* $(1/2, 1)$ *satisfying* $0 \le q_i'(u) < 1$, *and* $q_i''(u) \ge 0$, $i = 1, 2$;
(A_4) $g_i \in C^2(\mathbb{R})$ *be such that* $0 \le g_i'(u) < 1$ *and* $g_i''(u) \le 0$, $i = 1, 2$.

Then, there exists a monotone sequence $\{\alpha_n\}$ *of approximate solutions that converges quadratically to the unique solution of the problem* (1.1).

Proof. Let $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(t, u) = f(t, u) + \phi(t, u)$ so that $F_{uu}(t, u) \ge 0$. Using the generalized mean value theorem together with (A_2), (A_3) and (A_4), we obtain

$$f(t, u) \ge f(t, v) + F_u(t, v)(u - v) + \phi(t, v) - \phi(t, u), \tag{3.1}$$

$$g_i(u) \le g_i(v) + g_i'(v)(u - v), \quad u, v \in \mathbb{R}, \tag{3.2}$$

$$q_i(u) \ge q_i(v) + q_i'(v)(u - v), \quad u, v \in \mathbb{R}. \tag{3.3}$$

We set

$$\begin{aligned}
&\bar{F}(t, u, \alpha) = f(t, \alpha) + F_u(t, \alpha)(u - \alpha) + \phi(t, \alpha) - \phi(t, u),\\
&Q_i(u, \alpha) = q_i(\alpha) + q_i'(\alpha)(u - \alpha),\\
&\bar{g}_i(u(1/2), \alpha, \beta) = g_i(\alpha(1/2)) + g_i'(\beta(1/2))(u(1/2) - \alpha(1/2)),
\end{aligned}$$

and note that

$$\bar{F}_u(t, u, \alpha) < 0, \qquad 0 \le (\partial/\partial u) Q_i(u, \alpha) < 1, \qquad 0 \le (\partial/\partial u)\bar{g}_i(u(1/2), \alpha, \beta) < 1.$$

Now, we fix $\alpha = \alpha_0$ and consider the problem

$$\begin{cases}
u''(t) + \sigma u'(t) + \bar{F}(t, u, \alpha_0) = 0, \quad t \in [0, 1],\\
u(0) - \mu_1 u'(0) = \bar{g}_1(u(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_1(u(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(u(s), \alpha_0(s))\mathrm{d}s,\\
u(1) + \mu_2 u'(1) = \bar{g}_2(u(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_2(u(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(u(s), \alpha_0(s))\mathrm{d}s.
\end{cases} \tag{3.4}$$

As a first step, it will be shown that α_0, β are respectively lower and upper solutions of (3.4). Using (A_1) together with the fact that $\bar{F}(t, \alpha_0, \alpha_0) = f(t, \alpha_0)$, $\bar{g}_1(\alpha(1/2), \alpha_0, \beta) = g_i(\alpha_0(1/2))$ and $Q_i(\alpha_0, \alpha_0) = q_i(\alpha_0)$, we have

$$\alpha_0''(t) + \sigma\alpha_0'(t) + \bar{F}(t, \alpha_0, \alpha_0) = \alpha_0''(t) + \sigma\alpha_0'(t) + f(t, \alpha_0) \ge 0, \quad t \in [0, 1],$$

$$\begin{aligned}
\alpha_0(0) - \mu_1\alpha_0'(0) &\le g_1(\alpha_0(1/2)) + \int_0^{\frac{1}{2}-} q_1(\alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(\alpha_0(s))\mathrm{d}s\\
&= \bar{g}_1(\alpha_0(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_1(\alpha_0(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(\alpha_0(s), \alpha_0(s))\mathrm{d}s,\\
\alpha_0(1) + \mu_2\alpha_0'(1) &\le g_2(\alpha_0(1/2)) + \int_0^{\frac{1}{2}-} q_2(\alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_2(\alpha_0(s))\mathrm{d}s\\
&= \bar{g}_2(\alpha_0(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_2(\alpha_0(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(\alpha_0(s), \alpha_0(s))\mathrm{d}s
\end{aligned}$$

and

$$\beta''(t) + \sigma\beta'(t) + \bar{F}(t, \beta, \alpha_0) \le \beta''(t) + \sigma\beta'(t) + f(t, \beta) \le 0, \quad t \in [0, 1].$$

Moreover, there exists $c_0 \in (\alpha_0(1/2), \beta(1/2))$ and $c_1 \in (\alpha_0, \beta)$ so that

$$\begin{aligned} g_1(\beta(1/2)) - \bar{g}_1(\beta(1/2), \alpha_0, \beta) &= g_1(\beta(1/2)) - g_1(\alpha_0(1/2)) - g_1'(\beta(1/2))(\beta(1/2) - \alpha_0(1/2)) \\ &= [g_1'(c_0) - g_1'(\beta(1/2))](\beta(1/2) - \alpha_0(1/2)) \ge 0, \\ q_1(\beta(s)) - Q_1(\beta(s), \alpha_0(s)) &= q_1(\beta(s)) - q_1(\alpha_0(s)) - q_1'(\beta(s))(\beta(s) - \alpha_0(s)) \\ &= [q_1'(c_1) - q_1'(\beta(s))](\beta(s) - \alpha_0(s)) \ge 0 \end{aligned}$$

and consequently, we obtain

$$\begin{aligned} \beta(0) - \mu_1\beta'(0) &\ge g_1(\beta(1/2)) + \int_0^{\frac{1}{2}-} q_1(\beta(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 q_1(\beta(s))\mathrm{d}s \\ &\ge \bar{g}_1(\beta(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_1(\beta(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(\beta(s), \alpha_0(s))\mathrm{d}s. \end{aligned}$$

Similarly, we have

$$\beta(1) + \mu_2\beta'(1) \ge \bar{g}_2(\beta(1/2), \alpha_0, \beta) + \int_0^{\frac{1}{2}-} Q_2(\beta(s), \alpha_0(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(\beta(s), \alpha_0(s))\mathrm{d}s.$$

Thus we conclude that α_0 and β are respectively lower and upper solutions of (3.4). Hence, by Theorems 2.1 and 2.2, there exists the unique solution α_1 of (3.4) such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta(t), \quad t \in [0, 1].$$

Next, we consider

$$\begin{cases} u''(t) + \sigma u'(t) + \bar{F}(t, u, \alpha_1) = 0, \quad t \in [0, 1], \\ u(0) - \mu_1 u'(0) = \bar{g}_1(u(1/2), \alpha_1, \beta) + \int_0^{\frac{1}{2}-} Q_1(u(s), \alpha_1(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(u(s), \alpha_1(s))\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \bar{g}_2(u(1/2), \alpha_1, \beta) + \int_0^{\frac{1}{2}-} Q_2(u(s), \alpha_1(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(u(s), \alpha_1(s))\mathrm{d}s. \end{cases} \tag{3.5}$$

Using the earlier arguments, it can be shown that α_1 and β are lower and upper solutions of (3.5) respectively and by Theorems 2.1 and 2.2, there exists the unique solution α_2 of (3.5) such that

$$\alpha_1(t) \le \alpha_2(t) \le \beta(t), \quad t \in [0, 1].$$

Continuing this process successively yields a sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \le \alpha_1(t) \le \alpha_2(t) \le \cdots \le \alpha_n \le \beta(t), \quad t \in [0, 1],$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$\begin{cases} u''(t) + \sigma u'(t) + \bar{F}(t, u, \alpha_{n-1}) = 0, \quad t \in [0, 1], \\ u(0) - \mu_1 u'(0) = \bar{g}_1(u(1/2), \alpha_{n-1}, \beta) + \int_0^{\frac{1}{2}-} Q_1(u(s), \alpha_{n-1}(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_1(u(s), \alpha_{n-1}(s))\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \bar{g}_2(u(1/2), \alpha_{n-1}, \beta) + \int_0^{\frac{1}{2}-} Q_2(u(s), \alpha_{n-1}(s))\mathrm{d}s + \int_{\frac{1}{2}+}^1 Q_2(u(s), \alpha_{n-1}(s))\mathrm{d}s \end{cases}$$

and is given by

$$\begin{aligned}\alpha_n(t) = {} & \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\bar{g}_1(\alpha_n(1/2),\alpha_{n-1},\beta)+\int_0^{\frac{1}{2}-} Q_1(\alpha_n(s),\alpha_{n-1}(s))\mathrm{d}s\right. \\ & \left.+\int_{\frac{1}{2}+}^1 Q_1(\alpha_n(s),\alpha_{n-1}(s))\mathrm{d}s\right]+\frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\bar{g}_2(\alpha_n(1/2),\alpha_{n-1},\beta)\right. \\ & \left.+\int_0^{\frac{1}{2}-} Q_2(\alpha_n(s),\alpha_{n-1}(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 Q_2(\alpha_n(s),\alpha_{n-1}(s))\mathrm{d}s\right] \\ & +\int_0^1 G(t,s)\bar{F}(s,\alpha_n(s),\alpha_{n-1}(s))\mathrm{d}s. \qquad (3.6)\end{aligned}$$

Using the fact that [0, 1] is compact and the monotone convergence of the sequence $\{\alpha_n\}$ is pointwise, it follows that the convergence of the sequence is uniform. If $u(t)$ is the limit point of the sequence, taking the limit $n \to \infty$ in (3.6), we obtain

$$\begin{aligned}u(t) = {} & \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[g_1(u(1/2))+\int_0^{\frac{1}{2}-} q_1(u(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 q_1(u(s))\mathrm{d}s\right] \\ & +\frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[g_2(u(1/2))+\int_0^{\frac{1}{2}-} q_2(u(s))\mathrm{d}s+\int_{\frac{1}{2}+}^1 q_2(u(s))\mathrm{d}s\right] \\ & +\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s.\end{aligned}$$

Thus, $u(t)$ is a solution of (1.1). Now, we show that the convergence of the sequence is quadratic. For that we set $e_n(t) = (u(t) - \alpha_n(t)) \geq 0$, $t \in [0, 1]$. In view of (A_2), it follows by Taylor's theorem that

$$\begin{aligned}e_n''(t)+\sigma e_n'(t) &= u''+\sigma u'-(\alpha_n''+\sigma\alpha_n') = -f(t,u)+\bar{F}(t,\alpha_n,\alpha_{n-1}) \\ &= -f(t,u)+f(t,\alpha_{n-1})+F_u(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})+\phi(t,\alpha_{n-1})-\phi(t,\alpha_n) \\ &= -f_u(t,c_1)(u-\alpha_{n-1})-F_u(t,\alpha_{n-1})(u-\alpha_n)+F_u(t,\alpha_{n-1})(u-\alpha_{n-1}) \\ &\quad -\phi_u(t,c_2)(\alpha_n-\alpha_{n-1}) \\ &= [-f_u(t,c_1)+F_u(t,\alpha_{n-1})-\phi_u(t,c_2)]e_{n-1}+[-F_u(t,\alpha_{n-1})+\phi_u(t,c_2)]e_n \\ &= [-F_u(t,c_1)+F_u(t,\alpha_{n-1})+\phi_u(t,c_1)-\phi_u(t,c_2)]e_{n-1} \\ &\quad +[-F_u(t,\alpha_{n-1})+\phi_u(t,c_2)]e_n \\ &\geq [-F_u(t,u)+F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})-\phi_u(t,\alpha_n)]e_{n-1} \\ &\quad +[-F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})]e_n \\ &\geq [-F_{uu}(t,c_3)-\phi_{uu}(t,c_4)]e_{n-1}^2-f_u(t,\alpha_{n-1})e_n \\ &\geq -A_1\|e_{n-1}\|^2, \qquad (3.7)\end{aligned}$$

where $\alpha_{n-1} \leq c_1, c_3 \leq u, \alpha_{n-1} \leq c_2, c_4 \leq \alpha_n$, A is a bound on $\|F_{uu}\|$, B is a bound on $\|\phi_{uu}\|$ and $A_1 = A + B$. Further, we have

$$\begin{aligned}e_n(0)-\mu_1 e_n'(0) &= g_1(u(1/2))-\bar{g}_1(\alpha_n(1/2),\alpha_{n-1},\beta)+\int_0^{\frac{1}{2}-}[q_1(u(s))-Q_1(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s \\ &\quad +\int_{\frac{1}{2}+}^1[q_1(u(s))-Q_1(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s \\ &= g_1(u(1/2))-g_1(\alpha_{n-1}(1/2))-g_1'(\beta(1/2))(\alpha_n-\alpha_{n-1}) \\ &\quad +\int_0^{\frac{1}{2}-}[q_1(u(s))-q_1(\alpha_{n-1}(s))-q_1'(\alpha_{n-1}(s))(\alpha_n-\alpha_{n-1})]\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{1}{2}+}^{1}[q_1(u(s))-q_1(\alpha_{n-1}(s))-q_1'(\alpha_{n-1}(s))(\alpha_n-\alpha_{n-1})]\mathrm{d}s\\
&=\frac{1}{2}g_1''(\xi_1)e_{n-1}^2(1/2)+g_1'(\beta(1/2))e_n(1/2)+\int_0^{\frac{1}{2}-}\left[q_1'(\alpha_{n-1}(s))e_n(s)\right.\\
&\left.+\ \frac{1}{2}q_1''(\zeta_2)e_{n-1}^2(s)\right]\mathrm{d}s,+\int_{\frac{1}{2}+}^{1}\left[q_1'(\alpha_{n-1}(s))e_n(s)+\frac{1}{2}q_1''(\zeta_2)e_{n-1}^2(s)\right]\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
e_n(1)+\mu_2e_n'(1)&=g_2(u(1/2))-\bar{g}_2(\alpha_n(1/2),\alpha_{n-1},\beta)+\int_0^{\frac{1}{2}-}[q_2(u(s))-Q_2(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s\\
&+\int_{\frac{1}{2}+}^{1}[q_2(u(s))-Q_2(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s\\
&=\frac{1}{2}g_2''(\xi_3)e_{n-1}^2(1/2)+g_2'(\beta(1/2))e_n(1/2)+\int_0^{\frac{1}{2}-}\left[q_2'(\alpha_{n-1}(s))e_n(s)\right.\\
&\left.+\ \frac{1}{2}q_2''(\zeta_4)e_{n-1}^2(s)\right]\mathrm{d}s,+\int_{\frac{1}{2}+}^{1}\left[q_2'(\alpha_{n-1}(s))e_n(s)+\frac{1}{2}q_2''(\zeta_4)e_{n-1}^2(s)\right]\mathrm{d}s,
\end{aligned}
$$

where $\alpha_{n-1}\le\xi_1,\xi_2,\xi_3,\xi_4\le u$. In view of ($A_3$) and ($A_4$), there exist $\lambda_i<1,\lambda_i^*<1$, $M_i\ge0$ and $M_i^*\ge0$ such that $g_i'\le\lambda_i^*,q_i'\le\lambda_i,q_i''\le2M_i$ and $g_i''\le2M_i^*,i=1,2$. Letting $\lambda=\max\{\lambda_1,\lambda_2\}$, $\lambda^*=\max\{\lambda_1^*,\lambda_2^*\}$, $M^*=\max\{M_1^*,M_2^*\}$, and $M=\max\{M_1,M_2\}$, we get

$$
\begin{cases}
e_n(0)-\mu_1e_n'(0)\le M^*e_{n-1}^2(1/2)+\lambda^*e_n(1/2)+\lambda\left[\int_0^{\frac{1}{2}-}e_n(s)\mathrm{d}s+\int_{\frac{1}{2}+}^{1}e_n(s)\mathrm{d}s\right]\\
\qquad+M\left[\int_0^{\frac{1}{2}-}e_{n-1}^2(s)\mathrm{d}s+\int_{\frac{1}{2}+}^{1}e_{n-1}^2(s)\mathrm{d}s\right],\\
e_n(1)+\mu_2e_n'(1)\le M^*e_{n-1}^2(1/2)+\lambda^*e_n(1/2)+\lambda\left[\int_0^{\frac{1}{2}-}e_n(s)\mathrm{d}s+\int_{\frac{1}{2}+}^{1}e_n(s)\mathrm{d}s\right]\\
\qquad+M\left[\int_0^{\frac{1}{2}-}e_{n-1}^2(s)\mathrm{d}s+\int_{\frac{1}{2}+}^{1}e_{n-1}^2(s)\mathrm{d}s\right].
\end{cases}
\tag{3.8}
$$

Using the estimates (3.7) and (3.8), we obtain

$$
\begin{aligned}
e_n(t)=&\ \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\Bigg(g_1(u(1/2))-\bar{g}_1(\alpha_n(1/2),\alpha_{n-1},\beta)\\
&+\int_0^{\frac{1}{2}-}[q_1(u(s))-Q_1(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s+\int_{\frac{1}{2}+}^{1}[q_1(u(s))-Q_1(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s\Bigg)\\
&+\frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\Bigg(g_2(u(1/2))-\bar{g}_2(\alpha_n(1/2),\alpha_{n-1},\beta)\\
&+\int_0^{\frac{1}{2}-}[q_2(u(s))-Q_2(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s+\int_{\frac{1}{2}+}^{1}[q_2(u(s))-Q_2(\alpha_n(s),\alpha_{n-1}(s))]\mathrm{d}s\Bigg)\\
&+\int_0^1G(t,s)[f(s,u(s))-\bar{F}(t,\alpha_n,\alpha_{n-1})]\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\Bigg[M^* e_{n-1}^2(1/2)+\lambda^* e_n(1/2)\\
&\quad+\lambda\left(\int_0^{\frac{1}{2}-} e_n(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_n(s)\mathrm{d}s\right)+M\left(\int_0^{\frac{1}{2}-} e_{n-1}^2(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_{n-1}^2(s)\mathrm{d}s\right)\Bigg]\\
&\quad+\frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\Bigg[M^* e_{n-1}^2(1/2)+\lambda^* e_n(1/2)+\lambda\left(\int_0^{\frac{1}{2}-} e_n(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_n(s)\mathrm{d}s\right)\\
&\quad+M\left(\int_0^{\frac{1}{2}-} e_{n-1}^2(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_{n-1}^2(s)\mathrm{d}s\right)\Bigg]-\int_0^1 G(t,s)[e_n''(s)+\sigma e_n'(s)]\mathrm{d}s\\
&\le \Bigg[M^* e_{n-1}^2(1/2)+\lambda^* e_n(1/2)+\lambda\left(\int_0^{\frac{1}{2}-} e_n(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_n(s)\mathrm{d}s\right)\\
&\quad+M\left(\int_0^{\frac{1}{2}-} e_{n-1}^2(s)\mathrm{d}s+\int_{\frac{1}{2}+}^1 e_{n-1}^2(s)\mathrm{d}s\right)\Bigg]+A_1\|e_{n-1}\|^2\int_0^1 G(t,s)\mathrm{d}s\\
&\le M^*\|e_{n-1}\|^2+\lambda^*\|e_n\|+\lambda\|e_n\|+M\|e_{n-1}\|^2+A_2\|e_{n-1}\|^2\\
&= \lambda^{**}\|e_n\|+M^{**}\|e_{n-1}\|^2,
\end{aligned}
$$

where A_2 provides a bound on $A_1\int_0^1 G(t,s)$. We choose λ^* and λ so that $\lambda^{**}=\lambda^*+\lambda<1$ and $M^{**}=M^*+M+A_2$. Taking the maximum over [0, 1], we get

$$\|e_n\|\le\frac{M^{**}}{1-\lambda^{**}}\|e_{n-1}\|^2,$$

where $\|u\|=\max\{|u(t)|:t\in[0,1]\}$. This establishes the quadratic convergence of the sequence of iterates. □

4. Conclusions

The existence of the unique solution of the forced Duffing equation subject to discontinuous type integral boundary conditions is established by applying the method of quasilinearization. Several results of special interest can be recorded by varying the nonlinear forcing functions $f(t,u)$, $g_i(u)$ and $q_i(u)$ appropriately in (1.1). If we take $q_1(.)\equiv 0, q_2(.)\equiv 0$ in (1.1), our problem corresponds to the forced Duffing equation with three-point nonlinear boundary conditions. In the case $g_1(.)\equiv 0$, $g_2(.)\equiv 0$ in (1.1), our main result corresponds to the usual (continuous) type of integral boundary conditions and thereby recovers the result of [2] (Theorem 3.1) as a special case. By taking $\mu_1=0=\mu_2$ in (1.1), our problem reduces to the Dirichlet boundary value problem involving the forced Duffing equation with discontinuous type integral boundary conditions.

References

[1] B. Ahmad, A quasilinearization method for a class of integro-differential equations with mixed nonlinearities, Nonlinear Anal. RWA 7 (2006) 997–1004.

[2] B. Ahmad, A. Alsaedi, B. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, Nonlinear Anal. RWA (2007) (in press).

[3] B. Ahmad, Generalized quasilinearization for nonlinear three-point boundary value problems with nonlocal conditions, Sarajevo J. Math. 3(16) (2) (2007).

[4] B. Ahmad, J.J. Nieto, N. Shahzad, The Bellman–Kalaba–Lakshmikantham quasilinearization method for Neumann problems, J. Math. Anal. Appl. 257 (2001) 356–363.

[5] F.T. Akyildiz, K. Vajravelu, Existence, uniqueness, and quasilinearization results for nonlinear differential equations arising in viscoelastic fluid flow, Differ. Equ. Nonlinear Mech. (2006) Art. ID 71717, 9 pp. (electronic).

[6] R. Bellman, R. Kalaba, Quasilinearization and Nonlinear Boundary Value Problems, Elsevier, New York, 1965.

[7] A. Bouziani, N.E. Benouar, Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math. 15 (1998) 47–58.

[8] A. Cabada, J.J. Nieto, Quasilinearization and rate of convergence for higher order nonlinear periodic boundary value problems, J. Optim. Theory Appl. 108 (2001) 97–107.

[9] J.R. Cannon, The one-dimensional heat equation, in: Encyclopedia of Mathematics and its Applications, vol. 23, Addison-Wesley, Mento Park, CA, 1984.
[10] J.R. Cannon, S. Perez Esteva, J. Van Der Hoek, A Galerkin procedure for the diffusion equation subject to the specification of mass, SIAM J. Numer. Anal. 24 (1987) 499–515.
[11] Y.S. Choi, K.Y. Chan, A parabolic equation with nonlocal boundary conditions arising from electrochemistry, Nonlinear Anal. 18 (1992) 317–331.
[12] M. Denche, A.L. Marhoune, Mixed problem with integral boundary condition for a high order mixed type partial differential equation, J. Appl. Math. Stoch. Anal. 16 (2003) 69–79.
[13] M. El-Gebeily, D. O'Regan, A quasilinearization method for a class of second order singular nonlinear differential equations with nonlinear boundary conditions, Nonlinear Anal. Real World Appl. 8 (2007) 174–186.
[14] R.E. Ewing, T. Lin, A class of parameter estimation techniques for fluid flow in porous media, Adv. Water Resour. 14 (1991) 89–97.
[15] L. Formaggia, F. Nobile, A. Quarteroni, A. Veneziani, Multiscale modelling of the circulatory system: A preliminary analysis, Comput. Vis. Sci. 2 (1999) 75–83.
[16] N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, Differ. Uravn. 13 (1977) 294–304.
[17] A.V. Kartynnik, A three-point mixed problem with an integral condition with respect to the space variable for second-order parabolic equations, Differ. Equ. 26 (1990) 1160–1166.
[18] V. Lakshmikantham, A.S. Vatsala, Generalized Quasilinearization for Nonlinear Problems, Mathematics and its Applications, vol. 440, Kluwer Academic Publishers, Dordrecht, 1998.
[19] V. Lakshmikantham, A.S. Vatsala, Generalized quasilinearization versus Newton's method, Appl. Math. Comput. 164 (2005) 523–530.
[20] V.B. Mandelzweig, F. Tabakin, Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs, Comput. Phys. Comm. 141 (2001) 268–281.
[21] J.J. Nieto, Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions, Proc. Amer. Math. Soc. 125 (1997) 2599–2604.
[22] J.J. Nieto, A. Torres, A nonlinear biomathematical model for the study of intracranial aneurysms, J. Neurol. Sci. 177 (2000) 18–23.
[23] S. Nikolov, S. Stoytchev, A. Torres, J.J. Nieto, Biomathematical modeling and analysis of blood flow in an intracranial aneurysms, Neurol. Res. 25 (2003) 497–504.
[24] P. Shi, Weak solution to evolution problem with a nonlocal constraint, SIAM J. Anal. 24 (1993) 46–58.
[25] N.I. Yurchuk, A mixed problem with an integral condition for some parabolic equations, Differ. Uravn. 22 (1986) 1457–1463.