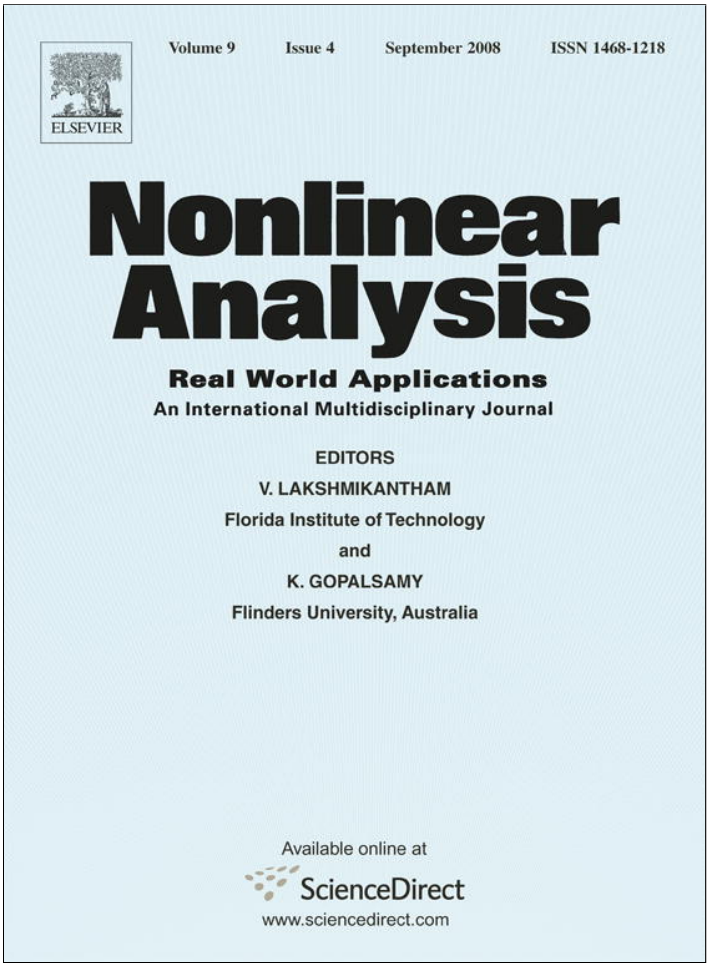
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Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions

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Abstract

A sequence of approximate solutions converging monotonically and quadratically to the unique solution of the forced Duffing equation with integral boundary conditions is obtained. We also establish the convergence of order k $(k \geqslant 2)$ for the sequence of iterates. The results obtained in this paper offer an algorithm to study the various practical phenomena such as prediction of the possible onset of vascular diseases, onset of chaos in speech, etc. Some interesting observations are presented.

MSC: 34B10; 34B15

Keywords: Duffing equation; Integral boundary conditions; Quasilinearization; Quadratic convergence; Higher order convergence

1. Introduction

Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, see for example [19,25,47,48]. Vascular diseases such as atherosclerosis and aneurysms are becoming frequent disorders in the industrialized world due to sedentary way of life and rich food. Causing more deaths than cancer, cardiovascular diseases are the leading cause of death in the world. In recent years, computational fluid dynamics (CFD) techniques have been used increasingly by researchers seeking to understand vascular hemodynamics. Most of the CFD-based hemodynamic studies so far have been conducted to represent *in vitro* conditions within restrictive assumptions. These studies under *in vitro* conditions are well suited to investigate basic phenomena related to fluid dynamics in vessels models but are not fully representative of actual patient hemodynamic conditions. In fact, CFD methods possess the potential to augment the data obtained from *in vitro* methods by providing a complete characterization of hemodynamic conditions (blood velocity and pressure as a function of space and time) under precisely controlled conditions. However, specific difficulties in CFD studies of blood flows are related to the boundary conditions. It is now recognized that the blood flow in a given district may depend on the global dynamics of the whole circulation. Consequently, it is sometimes necessary to couple the 3D blood flow solver to a low order model for the entire vascular system [26]. A second difficulty is related to the limitations of the existing *in vitro* anemometry techniques. Indeed, the space resolution is far too coarse to tackle even the largest scales

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of the blood flow details. As a consequence, the boundary conditions (e.g. the instantaneous velocity profile at the inlet section of the computed domain) are unknown for an *in vitro* blood flow computation. Most of the times, one assumes some analytical space–time evolution for prescribing the inlet profile. Taylor et al. [49] propose to assume very long circular vessel geometry upstream the inlet section so that the analytic solution of Womersley [51] can be prescribed. However, it is not always justified to assume a circular cross-section. In order to cope with this problem, an alternative approach prescribing integral boundary conditions is presented in Ref. [41]. The validity of this approach is verified by computing both steady and pulsated channel flows for Womersley number upto 15. For more details of boundary value problems involving integral boundary conditions, see for instance, [10,12,16–18,20,27,29,30,52] and references therein.

Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of Duffing equation is in the field of the prediction of diseases. A careful measurement and analysis of a strongly chaotic voice has the potential to serve as an early warning system for more serious chaos and possible onset of disease. This chaos is stimulated with the help of Duffing equation. In fact, the success at analyzing and predicting the onset of chaos in speech and its simulation by equations such as the Duffing equation has enhanced the hope that we might be able to predict the onset of arrhythmia and heart attacks someday. However, such predictions are based on the numerical solutions of the Duffing equation. One of the efficient analytic methods for solving boundary value problems is the monotone iterative technique. This technique coupled with the method of upper and lower solutions [8,21,28,32,43,44,50] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear [13,36]. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization (QSL) [11]. The nineties brought new dimensions to this technique when Lakshmikantham [34,35] generalized the method of QSL by relaxing the convexity assumption. This development was so significant that it attracted the attention of many researchers and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, for instance, see [1–7,9,14,15,22–24,31,33,37–40,42,45,46] and the references therein. In view of its diverse applications, this approach is quite an elegant and easier for application algorithms. To the best of our knowledge, the method of QSL has not been developed for Duffing equation with integral boundary conditions.

In this paper, we apply a QSL technique to obtain the analytic approximation of the solution of the forced Duffing equation with integral boundary conditions. In fact, we obtain a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem. We also discuss the rapid convergence of the sequence of iterates.

2. Preliminaries

Consider the following boundary value problem:

$$\begin{cases} u''(t)+\sigma u'(t)+f(t,u)=0, \quad 0<t<1, \quad \sigma\in\mathbb{R}-\{0\}, \\ u(0)-\mu_1 u'(0)=\int_0^1 q_1(u(s))\,\mathrm{d}s, \quad u(1)+\mu_2 u'(1)=\int_0^1 q_2(u(s))\,\mathrm{d}s, \end{cases} \tag{2.1}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $q_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are continuous functions and μ_i are nonnegative constants. Clearly the homogenous problem

$$u''(t)+\sigma u'(t)=0, \quad 0<t<1,$$

$$u(0)-\mu_1 u'(0)=0, \quad u(1)+\mu_2 u'(1)=0,$$

has only the trivial solution. Thus, for any $\rho,\theta_1,\theta_2\in C[0,1]$, the associated nonhomogeneous linear problem

$$u''(t)+\sigma u'(t)+\rho(t)=0, \quad 0<t<1,$$

$$u(0)-\mu_1 u'(0)=\int_0^1 \theta_1(s)\,\mathrm{d}s, \quad u(1)+\mu_2 u'(1)=\int_0^1 \theta_2(s)\,\mathrm{d}s,$$

has a unique solution $u(t)$ which, by Green's function method, can be written as

$$u(t)=G_1(t)+\int_0^1 G(t,s)\rho(s)\,\mathrm{d}s,$$

where $G_1(t)$ is the unique solution of the problem

$$u''(t)+\sigma u'(t)=0,\quad 0<t<1,$$
$$u(0)-\mu_1 u'(0)=\int_0^1 \theta_1(s)\,\mathrm{d}s,\quad u(1)+\mu_2 u'(1)=\int_0^1 \theta_2(s)\,\mathrm{d}s,$$

and is given by

$$G_1(t)=\frac{1}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\times\left[((-1+\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t})\int_0^1\theta_1(s)\,\mathrm{d}s+((1+\sigma\mu_1)-\mathrm{e}^{-\sigma t})\int_0^1\theta_2(s)\,\mathrm{d}s\right],$$

and

$$G(t,s)=\Lambda\begin{cases}[(1-\sigma\mu_2)-\mathrm{e}^{\sigma(1-s)}][(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}], & 0\leqslant t\leqslant s,\\ [(1-\sigma\mu_2)-\mathrm{e}^{\sigma(1-t)}][(1+\sigma\mu_1)-\mathrm{e}^{-\sigma s}], & s\leqslant t\leqslant 1,\end{cases}$$

$$\Lambda=\frac{\mathrm{e}^{\sigma s}}{\sigma[(1-\sigma\mu_2)-(1+\sigma\mu_1)\mathrm{e}^{\sigma}]}.$$

We note that $G(t,s)>0$ on $(0,1)\times(0,1)$.

Definition 2.1. A function $\alpha\in C^2[0,1]$ is a lower solution of (2.1) if

$$\alpha''(t)+\sigma\alpha'(t)+f(t,\alpha(t))\geqslant 0,\quad 0<t<1,$$
$$\alpha(0)-\mu_1\alpha'(0)\leqslant\int_0^1 q_1(\alpha(s))\,\mathrm{d}s,\quad \alpha(1)+\mu_2\alpha'(1)\leqslant\int_0^1 q_2(\alpha(s))\,\mathrm{d}s.$$

Similarly, $\beta\in C^2[0,1]$ is an upper solution of (2.1) if the inequalities in the definition of lower solution are reversed.

Theorem 2.1. *Let α and β be lower and upper solutions of the boundary value problem* (2.1), *respectively. Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be such that $f_u(t,u)<0$ and $q_i:\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying a one sided Lipschitz condition*: $q_i(u)-q_i(v)\leqslant L_i(u-v)$, $0\leqslant L_i<1$, $i=1,2$. *Then $\alpha(t)\leqslant\beta(t)$, $t\in[0,1]$.*

Proof. Set $x(t)=\alpha(t)-\beta(t)$, $t\in[0,1]$ so that

$$\begin{cases}x(0)-\mu_1x'(0)\leqslant\int_0^1[q_1(\alpha(s))-q_1(\beta(s))]\,\mathrm{d}s,\\ x(1)+\mu_2x'(1)\leqslant\int_0^1[q_2(\alpha(s))-q_2(\beta(s))]\,\mathrm{d}s.\end{cases}\tag{2.2}$$

For the sake of contradiction, suppose that $x(t)>0$ for $t\in[0,1]$. Then $x(t)$ has a positive maximum at some $t_0\in[0,1]$. If $t_0\in(0,1)$, then $x(t_0)>0$, $x'(t_0)=0$ and $x''(t_0)\leqslant 0$. In view of the decreasing property of the function $f(t,u)$ in u, it follows that

$$x''(t_0)+\sigma x'(t_0)=\alpha''(t_0)+\sigma\alpha'(t_0)-(\beta''(t_0)+\sigma\beta'(t_0))\geqslant -f(t_0,\alpha(t_0))+f(t_0,\beta(t_0))>0,$$

which is a contradiction. If $t_0=0$, then $x(0)>0$, $x'(0)=0$. Using (2.2) together with the assumption that q_1 satisfies a one sided Lipschitz condition, we obtain the following contradiction:

$$x(0)=x(0)-\mu_1x'(0)\leqslant\int_0^1[q_1(\alpha(s))-q_1(\beta(s))]\,\mathrm{d}s\leqslant L_1\max_{t\in[0,1]}x(t)=L_1x(0)<x(0).$$

A similar contradiction occurs for $t_0=1$. Hence $\alpha(t)\leqslant\beta(t)$, $t\in[0,1]$. □

Theorem 2.2. *Assume that α and β are lower and upper solutions of the boundary value problem* (2.1), *respectively, such that $\alpha(t) \leqslant \beta(t)$, $t \in [0, 1]$. If $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $q_i : \mathbb{R} \to \mathbb{R}$ are continuous and q_i satisfy a one sided Lipschitz condition, then there exists a solution $u(t)$ of* (2.1) *such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [0, 1]$.*

Proof. Let us define $F(t, u)$ and $Q_i(u)$ by

$$F(t,u) = \begin{cases} f(t,\beta(t)) - \dfrac{u-\beta(t)}{1+|u-\beta|} & \text{if } u > \beta, \\ f(t,u) & \text{if } \alpha \leqslant u \leqslant \beta, \\ f(t,\alpha(t)) - \dfrac{u-\alpha(t)}{1+|u-\alpha|} & \text{if } u < \alpha \end{cases}$$

and

$$Q_i(u) = \begin{cases} q_i(\beta(t)) & \text{if } u > \beta, \\ q_i(u(t)) & \text{if } \alpha \leqslant u \leqslant \beta, \\ q_i(\alpha(t)) & \text{if } u < \alpha. \end{cases}$$

Since $F(t, u)$ and $Q_i(u)$ are continuous and bounded, it follows that there exists a solution $u(t)$ of the problem

$$\begin{cases} u''(t) + \sigma u'(t) + F(t,u) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s))\,\mathrm{d}s, \quad u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s))\,\mathrm{d}s. \end{cases} \tag{2.3}$$

In relation to (2.3), we have

$$\alpha''(t) + \sigma\alpha'(t) + F(t,\alpha(t)) = \alpha''(t) + \sigma\alpha'(t) + f(t,\alpha(t)) \geqslant 0, \quad 0 < t < 1,$$

$$\alpha(0) - \mu_1\alpha'(0) \leqslant \int_0^1 q_1(\alpha(s))\,\mathrm{d}s = \int_0^1 Q_1(\alpha(s))\,\mathrm{d}s,$$

$$\alpha(1) + \mu_2\alpha'(1) \leqslant \int_0^1 q_2(\alpha(s))\,\mathrm{d}s = \int_0^1 Q_2(\alpha(s))\,\mathrm{d}s$$

and

$$\beta''(t) + \sigma\beta'(t) + F(t,\beta(t)) = \beta''(t) + \sigma\beta'(t) + f(t,\beta(t)) \leqslant 0, \quad 0 < t < 1,$$

$$\beta(0) - \mu_1\beta'(0) \geqslant \int_0^1 q_1(\beta(s))\,\mathrm{d}s = \int_0^1 Q_1(\beta(s))\,\mathrm{d}s,$$

$$\beta(1) + \mu_2\beta'(1) \geqslant \int_0^1 q_2(\beta(s))\,\mathrm{d}s = \int_0^1 Q_2(\beta(s))\,\mathrm{d}s,$$

which imply that α and β are lower and upper solutions of (2.3), respectively. By definition of $F(t, u)$, it follows that any solution $u \in [\alpha, \beta]$ of (2.3) is indeed a solution of (2.1). Thus, we just need to show that any solution $u(t)$ of (2.3) satisfies $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [0, 1]$. Let us assume that $\alpha(t) > u(t)$ on $[0, 1]$. Then the function $y(t) = \alpha(t) - u(t)$ has a positive maximum at some $t = t_0 \in [0, 1]$. If $t_0 \in (0, 1)$, then $y(t_0) > 0$, $y'(t_0) = 0$, $y''(t_0) \leqslant 0$. On the other hand,

$$\begin{aligned} y''(t_0) + \sigma y'(t_0) &= \alpha''(t_0) + \sigma\alpha'(t_0) - [u''(t_0) + \sigma u'(t_0)] \\ &\geqslant -F(t_0, \alpha(t_0)) + F(t_0, u(t_0)) \\ &= -f(t_0, \alpha(t_0)) + f(t_0, \alpha(t_0)) - \frac{u - \alpha(t_0)}{1 + |u - \alpha_0|} > 0, \end{aligned}$$

which contradicts our assumption. If $t_0 = 0$, then $y(0) > 0$, $y'(0) = 0$ and

$$\begin{aligned} y(0) = \alpha(0) - u(0) &\leqslant \mu_1 y'(0) + \int_0^1 [q_1(\alpha(s)) - Q_1(u(s))]\,\mathrm{d}s \\ &= \int_0^1 [q_1(\alpha(s)) - Q_1(u(s))]\,\mathrm{d}s. \end{aligned}$$

If $u(t) < \alpha(t)$, then $Q_1(u(s)) = q_1(\alpha(s))$ and consequently we have the contradiction $y(0) \leqslant 0$. If $u(t) > \beta(t)$, then $Q_1(u(s)) = q_1(\beta(s))$. Hence, in view of the fact that q_1 satisfies a one sided Lipschitz condition, we have $q_1(\alpha(s)) - Q_1(u(s)) = (q_1(\alpha(s)) - q_1(\beta(s))) \leqslant L_1(\alpha(s) - \beta(s))$ so that $y(0) \leqslant L_1 \max_{t\in[0,1]}(\alpha(t) - \beta(t)) = L_1(\alpha(0) - \beta(0)) \leqslant 0$ which is again a contradiction. For $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, we also get the contradiction $y(0) \leqslant 0$. In a similar manner, $t_0 = 1$ yields a contradiction. Thus, $\alpha(t) \leqslant u(t)$, $t \in [0, 1]$. On the same pattern, it can be shown that $u(t) \leqslant \beta(t)$, $t \in [0, 1]$. Hence we conclude that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$, $t \in [0, 1]$. □

Corollary 2.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be such that $f_u(t, u) < 0$ and $q_i : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying a one sided Lipschitz condition. Then the solution of* (2.1) *is unique.*

3. Main results

Theorem 3.1. *Assume that*

(A_1) *α and $\beta \in C^2[0, 1]$ are, respectively, lower and upper solutions of* (2.1) *such that $\alpha(t) \leqslant \beta(t)$, $t \in [0, 1]$;*
(A_2) *$f(t, u) \in C^2([0, 1] \times \mathbb{R})$ be such that $f_u(t, u) < 0$ and $(f_{uu}(t, u) + \phi_{uu}(t, u)) \geqslant 0$, where $\phi_{uu}(t, u) \geqslant 0$ for some continuous function $\phi(t, u)$ on $[0, 1] \times \mathbb{R}$;*
(A_3) *$q_i \in C^2(\mathbb{R})$ be such that $0 \leqslant q_i'(u) < 1$, and $q_i''(u) \geqslant 0$, $i = 1, 2$.*

Then, there exists a sequence $\{\alpha_n\}$ of approximate solutions converging monotonically and quadratically to the unique solution of the problem (2.1).

Proof. Let $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(t, u) = f(t, u) + \phi(t, u)$ so that $F_{uu}(t, u) \geqslant 0$. Using the generalized mean value theorem together with (A_2) and (A_3), we obtain

$$f(t, u) \geqslant f(t, v) + F_u(t, v)(u - v) + \phi(t, v) - \phi(t, u), \tag{3.1}$$

$$q_i(u) \geqslant q_i(v) + q_i'(v)(u - v), \quad u, v \in \mathbb{R}. \tag{3.2}$$

We set

$$g(t, u, v) = f(t, v) + F_u(t, v)(u - v) + \phi(t, v) - \phi(t, u), \tag{3.3}$$

and note that $g_u(t, u, v) = [F_u(t, v) - \phi_u(t, u)] \leqslant [F_u(t, u) - \phi_u(t, u)] = f_u(t, u) < 0$ with

$$\begin{cases} f(t, u) \geqslant g(t, u, v), \\ f(t, u) = g(t, u, u). \end{cases} \tag{3.4}$$

Let us define

$$Q_i(u, v) = q_i(v) + q_i'(v)(u - v), \tag{3.5}$$

so that $0 \leqslant (\partial/\partial u)Q_i(u, v) = q_i' < 1$ and

$$\begin{cases} q_i(u) \geqslant Q_i(u, v), \\ q_i(u) = Q_i(u, u). \end{cases} \tag{3.6}$$

Now, we fix $\alpha_0 = \alpha$ and consider the problem

$$\begin{cases} u''(t) + \sigma u'(t) + g(t, u, \alpha_0) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s), \alpha_0(s))\, ds, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s), \alpha_0(s))\, ds. \end{cases} \tag{3.7}$$

Using (A_1), (3.4) and (3.6), we obtain

$$\alpha_0''(t) + \sigma\alpha_0'(t) + g(t, \alpha_0, \alpha_0) = \alpha_0''(t) + \sigma\alpha_0'(t) + f(t, \alpha_0) \geqslant 0, \quad 0 < t < 1,$$

$$\alpha_0(0) - \mu_1\alpha_0'(0) \leqslant \int_0^1 q_1(\alpha_0(s))\,\mathrm{d}s = \int_0^1 Q_1(\alpha_0(s), \alpha_0(s))\,\mathrm{d}s,$$

$$\alpha_0(1) + \mu_2\alpha_1'(0) \leqslant \int_0^1 q_2(\alpha_0(s))\,\mathrm{d}s = \int_0^1 Q_2(\alpha_0(s), \alpha_0(s))\,\mathrm{d}s$$

and

$$\beta''(t) + \sigma\beta'(t) + g(t, \beta, \alpha_0) \leqslant \beta''(t) + \sigma\beta'(t) + f(t, \beta) \leqslant 0, \quad 0 < t < 1,$$

$$\beta(0) - \mu_1\beta'(0) \geqslant \int_0^1 q_1(\beta(s))\,\mathrm{d}s \geqslant \int_0^1 Q_1(\beta(s), \alpha_0(s))\,\mathrm{d}s,$$

$$\beta(1) + \mu_2\beta'(1) \geqslant \int_0^1 q_2(\beta(s))\,\mathrm{d}s \geqslant \int_0^1 Q_2(\beta(s), \alpha_0(s))\,\mathrm{d}s,$$

which imply that α_0 and β are, respectively, lower and upper solutions of (3.7). It follows by Theorems 2.1 and 2.2 that there exists a unique solution α_1 of (3.7) such that

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta(t), \quad t \in [0, 1].$$

Next, we consider

$$\begin{cases} u''(t) + \sigma u'(t) + g(t, u, \alpha_1) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s), \alpha_1(s))\,\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s), \alpha_1(s))\,\mathrm{d}s. \end{cases} \tag{3.8}$$

Using the earlier arguments, it can be shown that α_1 and β are lower and upper solutions of (3.8), respectively and hence by Theorems 2.1 and 2.2, there exists a unique solution α_2 of (3.8) such that

$$\alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta(t), \quad t \in [0, 1].$$

Continuing this process successively yields a sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \alpha_2(t) \leqslant \cdots \leqslant \alpha_n \leqslant \beta(t), \quad t \in [0, 1],$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$u''(t) + \sigma u'(t) + g(t, u, \alpha_{n-1}) = 0, \quad 0 < t < 1,$$

$$u(0) - \mu_1 u'(0) = \int_0^1 Q_1(u(s), \alpha_{n-1}(s))\,\mathrm{d}s,$$

$$u(1) + \mu_2 u'(1) = \int_0^1 Q_2(u(s), \alpha_{n-1}(s))\,\mathrm{d}s,$$

and is given by

$$\begin{aligned} \alpha_n(t) = {} & \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma} + \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}} \int_0^1 Q_1(\alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s \\ & + \frac{(1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}} \int_0^1 Q_2(\alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s \\ & + \int_0^1 G(t, s) g(s, \alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Using the fact that [0, 1] is compact and the monotone convergence of the sequence $\{\alpha_n\}$ is pointwise, it follows by the standard arguments (Arzela Ascoli convergence criterion, Dini's theorem [32,38]) that the convergence of the sequence is uniform. If $u(t)$ is the limit point of the sequence, taking the limit $n \to \infty$ in (3.9), we obtain

$$u(t) = \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \int_0^1 q_1(u(s))\,ds + \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \int_0^1 q_2(u(s))\,ds + \int_0^1 G(t,s)f(s,u(s))\,ds.$$

Thus, $u(t)$ is a solution of (2.1). Now, we show that the convergence of the sequence is quadratic. For that we set $e_n(t) = (u(t) - \alpha_n(t)) \geqslant 0$, $t \in [0, 1]$. In view of (A_2) and (3.3), it follows by Taylor's theorem that

$$\begin{aligned}
e_n''(t) + \sigma e_n'(t) &= u'' + \sigma u' - (\alpha_n'' + \sigma\alpha_n') = -f(t,u) + g(t,\alpha_n,\alpha_{n-1}) \\
&= -f(t,u) + f(t,\alpha_{n-1}) + F_u(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1}) + \phi(t,\alpha_{n-1}) - \phi(t,\alpha_n) \\
&= -f_u(t,c_1)(u-\alpha_{n-1}) - F_u(t,\alpha_{n-1})(u-\alpha_n) + F_u(t,\alpha_{n-1})(u-\alpha_{n-1}) \\
&\quad - \phi_u(t,c_2)(\alpha_n - \alpha_{n-1}) \\
&= [-f_u(t,c_1) + F_u(t,\alpha_{n-1}) - \phi_u(t,c_2)]e_{n-1} + [-F_u(t,\alpha_{n-1}) + \phi_u(t,c_2)]e_n \\
&= [-F_u(t,c_1) + F_u(t,\alpha_{n-1}) + \phi_u(t,c_1) - \phi_u(t,c_2)]e_{n-1} \\
&\quad + [-F_u(t,\alpha_{n-1}) + \phi_u(t,c_2)]e_n \\
&\geqslant [-F_u(t,u) + F_u(t,\alpha_{n-1}) + \phi_u(t,\alpha_{n-1}) - \phi_u(t,\alpha_n)]e_{n-1} \\
&\quad + [-F_u(t,\alpha_{n-1}) + \phi_u(t,\alpha_{n-1})]e_n \\
&= [-F_{uu}(t,c_3) - \phi_{uu}(t,c_4)]e_{n-1}^2 - f_u(t,\alpha_{n-1})e_n \\
&\geqslant -[A+B]e_{n-1}^2 \\
&= -M\|e_{n-1}\|^2,
\end{aligned} \tag{3.10}$$

where $\alpha_{n-1} \leqslant c_1, c_3 \leqslant u$, $\alpha_{n-1} \leqslant c_2, c_4 \leqslant \alpha_n$, A is a bound on $\|F_{uu}\|$, B is a bound on $\|\phi_{uu}\|$ for $t \in (0, 1)$ and $M = A + B$. Further, in view of (3.5), we have

$$\begin{aligned}
e_n(0) - \mu_1 e_n'(0) &= \int_0^1 [q_1(u(s)) - Q_1(\alpha_n(s), \alpha_{n-1}(s))]\,ds \\
&= \int_0^1 [q_1(u(s)) - q_1(\alpha_{n-1}(s)) - q_1'(\alpha_{n-1}(s))(\alpha_n - \alpha_{n-1})]\,ds \\
&= \int_0^1 \left[q_1'(\alpha_{n-1}(s))e_n(s) + \frac{1}{2}q_2''(\zeta_1)e_{n-1}^2(s)\right] ds, \\
e_n(1) + \mu_2 e_n'(1) &= \int_0^1 [q_2(u(s)) - Q_2(\alpha_n(s), \alpha_{n-1}(s))]\,ds \\
&= \int_0^1 \left[q_2'(\alpha_{n-1}(s))e_n(s) + \frac{1}{2}q_2''(\zeta_2)e_{n-1}^2(s)\right] ds,
\end{aligned}$$

where $\alpha_{n-1} \leqslant \zeta_1, \zeta_2 \leqslant u$. In view of ($A_3$), there exist $\lambda_i < 1$ and $M_i \geqslant 0$ such that $q_i'(\alpha_{n-1}(s)) \leqslant \lambda_i$ and $\frac{1}{2}q_i''(\zeta_i) \leqslant M_i$ $(i = 1, 2)$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$ and $M_3 = \max\{M_1, M_2\}$, then

$$\begin{cases} e_n(0) - \mu_1 e_n'(0) \leqslant \lambda \int_0^1 e_n(s)\,ds + M_3 \int_0^1 e_{n-1}^2(s)\,ds, \\ e_n(1) + \mu_2 e_n'(1) \leqslant \lambda \int_0^1 e_n(s)\,ds + M_3 \int_0^1 e_{n-1}^2(s)\,ds. \end{cases} \tag{3.11}$$

Using the estimates (3.10) and (3.11), we obtain

$$\begin{aligned}
e_n(t) &= \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\int_0^1 [q_1(u(s)) - Q_1(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\
&\quad + \frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\int_0^1 [q_2(u(s)) - Q_2(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\
&\quad + \int_0^1 G(t,s)[f(s,u(s)) - g(t,\alpha_n,\alpha_{n-1})]\,\mathrm{d}s \\
&\leqslant \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}+\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\lambda\int_0^1 e_n(s)\,\mathrm{d}s + M_3\int_0^1 e_{n-1}^2(s)\,\mathrm{d}s\right] \\
&\quad + \frac{(1+\sigma\mu_1)-\mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\lambda\int_0^1 e_n(s)\,\mathrm{d}s + M_3\int_0^1 e_{n-1}^2(s)\,\mathrm{d}s\right] \\
&\quad - \int_0^1 G(t,s)[e_n''(s)+\sigma e_n'(s)]\,\mathrm{d}s \\
&\leqslant \lambda\int_0^1 e_n(s)\,\mathrm{d}s + M_3\int_0^1 e_{n-1}^2(s)\,\mathrm{d}s + M\|e_{n-1}\|^2\int_0^1 G(t,s)\,\mathrm{d}s \\
&\leqslant \lambda\|e_n\| + M_3\|e_{n-1}\|^2 + M_4\|e_{n-1}\|^2 = \lambda\|e_n\| + M_5\|e_{n-1}\|^2,
\end{aligned}$$

where M_4 provides a bound on $M\int_0^1 G(t,s)$ and $M_5 = M_4 + M_3$. Taking the maximum over [0, 1], we get

$$\|e_n\| \leqslant \frac{M_5}{1-\lambda}\|e_{n-1}\|^2,$$

where $\|u\| = \{|u(t)| : t \in [0,1]\}$. This establishes the quadratic convergence of the sequence of iterates. □

Theorem 3.2 (*Higher order convergence*). *Assume that*

(B$_1$) *α and $\beta \in C^2[0,1]$ are, respectively, lower and upper solutions of* (2.1) *such that $\alpha(t) \leqslant \beta(t)$, $t \in [0,1]$;*
(B$_2$) *$f(t,u) \in C^k([0,1]\times\mathbb{R})$ be such that $(\partial^p/\partial u^p) f_u < 0$ $(p=1,2,3,\ldots,k-1)$ and $(\partial^k/\partial u^k)(f(t,u)+\phi(t,u)) \geqslant 0$ with $(\partial^k/\partial u^k)\phi(t,u) \geqslant 0$ for some continuous function $\phi(t,u)$ on $C^k[[0,1]\times\mathbb{R}]$;*
(B$_3$) *$q_j \in C^k(\mathbb{R})$ be such that $(d^i/du^i)q_j(u) \leqslant M/(\beta-\alpha)^{i-1}$ $(i=1,2,\ldots,k-1,\ j=1,2)$ and $(d^k/du^k)q_j(u) \geqslant 0$, where $M < \frac{1}{3}$.*

Then, there exists a monotone sequence $\{\alpha_n\}$ of approximate solutions converging uniformly and rapidly to the unique solution of the problem (2.1) *with the order of convergence k $(k \geqslant 2)$.*

Proof. Using Taylor's theorem and the assumptions (B$_2$) and (B$_3$), we obtain

$$f(t,u) \geqslant \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v)\frac{(u-v)^i}{i!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^k}{k!} \tag{3.12}$$

and

$$q_j(u) \geqslant \sum_{i=0}^{k-1} \frac{\mathrm{d}^i}{\mathrm{d}u^i} q_j(v)\frac{(u-v)^i}{i!} \tag{3.13}$$

where $\alpha \leqslant v \leqslant \xi \leqslant u \leqslant \beta$. We set

$$h(t,u,v) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v) \frac{(u-v)^i}{i!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^k}{k!}, \tag{3.14}$$

$$Q_j^*(u,v) = \sum_{i=0}^{k-1} \frac{\mathrm{d}^i}{\mathrm{d}u^i} q_j(v) \frac{(u-v)^i}{i!}. \tag{3.15}$$

Observe that $h(t,u,v)$ and $Q_j^*(u,v)$ are continuous, bounded and satisfy the following relations:

$$\begin{cases} f(t,u) \geqslant h(t,u,v), \\ f(t,u) = h(t,u,u), \end{cases} \tag{3.16}$$

$$\begin{cases} q_j(u) \geqslant Q_j^*(u,v), \\ q_j(u) = Q_j^*(u,u), \end{cases} \tag{3.17}$$

$$h_u(t,u,v) = \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t,v) \frac{(u-v)^{i-1}}{(i-1)!} - \frac{\partial^k}{\partial u^k}\phi(t,\xi)\frac{(u-v)^{k-1}}{(k-1)!} \leqslant 0,$$

$$\begin{aligned} \frac{\partial}{\partial u} Q_j^*(u,v) = \sum_{i=1}^{k-1} \frac{\mathrm{d}^i}{\mathrm{d}u^i} q_j(v) \frac{(u-v)^{i-1}}{(i-1)!} &\leqslant \sum_{i=1}^{k-1} \frac{M}{(\beta-\alpha)^{i-1}} \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\ &\leqslant M\left(3 - \frac{1}{2^{k-2}}\right) < 1. \end{aligned}$$

Letting $\alpha_0 = \alpha$, we consider the problem

$$\begin{cases} u''(t) + \sigma u'(t) + h(t,u,\alpha_0) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \int_0^1 Q_1^*(u(s),\alpha_0(s))\,\mathrm{d}s, \\ u(1) + \mu_2 u'(1) = \int_0^1 Q_2^*(u(s),\alpha_0(s))\,\mathrm{d}s. \end{cases} \tag{3.18}$$

Using (B_1), (3.16) and (3.17), we obtain

$$\alpha_0''(t) + \sigma\alpha_0'(t) + h(t,\alpha_0,\alpha_0) = \alpha_0''(t) + \sigma\alpha_0'(t) + f(t,\alpha_0) \geqslant 0, \quad 0 < t < 1,$$

$$\alpha_0(0) - \mu_1\alpha_0'(0) \leqslant \int_0^1 q_1(\alpha_0(s))\,\mathrm{d}s = \int_0^1 Q_1^*(\alpha_0(s),\alpha_0(s))\,\mathrm{d}s,$$

$$\alpha_0(1) + \mu_2\alpha_1'(0) \leqslant \int_0^1 q_2(\alpha_0(s))\,\mathrm{d}s = \int_0^1 Q_2^*(\alpha_0(s),\alpha_0(s))\,\mathrm{d}s$$

and

$$\beta''(t) + \sigma\beta'(t) + h(t,\beta,\alpha_0) \leqslant \beta''(t) + \sigma\beta'(t) + f(t,\beta) \leqslant 0, \quad 0 < t < 1,$$

$$\beta(0) - \mu_1\beta'(0) \geqslant \int_0^1 q_1(\beta(s))\,\mathrm{d}s \geqslant \int_0^1 Q_1^*(\beta(s),\alpha_0(s))\,\mathrm{d}s,$$

$$\beta(1) + \mu_2\beta'(1) \geqslant \int_0^1 q_2(\beta(s))\,\mathrm{d}s \geqslant \int_0^1 Q_2^*(\beta(s),\alpha_0(s))\,\mathrm{d}s.$$

Thus, it follows by definition that α_0 and β are, respectively, lower and upper solutions of (3.18). As before, by Theorems 2.1 and 2.2, there exists a unique solution α_1 of (3.18) such that

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta(t), \quad t \in [0,1].$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \alpha_2(t) \leqslant \cdots \leqslant \alpha_n \leqslant \beta(t), \quad t \in [0,1],$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$u''(t) + \sigma u'(t) + h(t, u, \alpha_{n-1}) = 0, \quad 0 < t < 1,$$
$$u(0) - \mu_1 u'(0) = \int_0^1 Q_1^*(u(s), \alpha_{n-1}(s))\,\mathrm{d}s,$$
$$u(1) + \mu_2 u'(1) = \int_0^1 Q_2^*(u(s), \alpha_{n-1}(s))\,\mathrm{d}s,$$

and is given by

$$\begin{aligned}\alpha_n(t) = {} & \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma} + \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}} \int_0^1 Q_1^*(\alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s \\ & + \frac{(1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}} \int_0^1 Q_2^*(\alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s \\ & + \int_0^1 G(t,s) h(s, \alpha_n(s), \alpha_{n-1}(s))\,\mathrm{d}s.\end{aligned}$$

Employing the arguments used in the proof of Theorem 3.1, we conclude that the sequence $\{\alpha_n\}$ converges uniformly to the unique solution $u(t)$ of (2.1).

In order to prove that the convergence of the sequence is of order $k (k \geqslant 2)$, we set $e_n(t) = u(t) - \alpha_n(t)$ and $a_n(t) = \alpha_{n+1}(t) - \alpha_n(t)$, $t \in [0, 1]$ and note that

$$e_n(t) \geqslant 0, \quad a_n(t) \geqslant 0, \quad e_{n+1}(t) = e_n(t) - a_n(t), \quad e_n^k \geqslant a_n^k.$$

Using Taylor's theorem, we find that

$$\begin{aligned} e_n''(t) + \sigma e_n'(t) &= u''(t) + \sigma u'(t) - (\alpha_n''(t) + \sigma \alpha_n'(t)) \\ &= -f(t, u(t)) + h(t, \alpha_n, \alpha_{n-1}) \\ &= -f(t, u(t)) + \sum_{i=0}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{a_{n-1}^i}{i!} - \frac{\partial^k}{\partial u^k} \phi(t, \xi) \frac{a_{n-1}^k}{k!} \\ &= -f(t, u(t)) + f(t, \alpha_{n-1}(t)) + \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{a_{n-1}^i}{i!} - \frac{\partial^k}{\partial u^k} \phi(t, \xi) \frac{a_{n-1}^k}{k!} \\ &= -f(t, u(t)) + f(t, \alpha_{n-1}(t)) + \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{e_{n-1}^i}{i!} \\ &\quad - \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{(e_{n-1}^i - a_{n-1}^i)}{i!} - \frac{\partial^k}{\partial u^k} \phi(t, \xi) \frac{a_{n-1}^k}{k!} \\ &= -\frac{\partial^k}{\partial u^k} f(t, \zeta) \frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{(e_{n-1} - a_{n-1})}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l \\ &\quad - \frac{\partial^k}{\partial u^k} \phi(t, \xi) \frac{a_{n-1}^k}{k!} \\ &\geqslant -\left(\frac{\partial^k}{\partial u^k} f(t, \xi) + \frac{\partial^k}{\partial u^k} \phi(t, \xi) \right) \frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{e_n}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l \\ &= -\frac{\partial^k}{\partial u^k} F(t, \xi)) \frac{e_{n-1}^k}{k!} - \sum_{i=1}^{k-1} \frac{\partial^i}{\partial u^i} f(t, \alpha_{n-1}) \frac{e_n}{i!} \sum_{l=0}^{i-1} e_{n-1}^{i-1-l} a_{n-1}^l \\ &\geqslant -N \frac{\|e_{n-1}\|^k}{k!}, \end{aligned} \tag{3.19}$$

where N is a bound for $(\partial^k/\partial u^k)F(t,\xi)$. Again, by Taylor's theorem and using (3.15), we obtain

$$\begin{aligned} q_j(u(s)) - Q_j^*(\alpha_n(s),\alpha_{n-1}(s)) &= \sum_{i=0}^{k-1}\frac{\mathrm{d}^i}{\mathrm{d}u^i}q_j(\alpha_{n-1})\frac{(u-\alpha_{n-1})^i}{i!} + \frac{\mathrm{d}^k}{\mathrm{d}u^k}q_j(c)\frac{(u-\alpha_{n-1})^k}{k!} \\ &\quad - \sum_{i=0}^{k-1}\frac{\mathrm{d}^i}{\mathrm{d}u^i}q_j(\alpha_{n-1})\frac{(\alpha_n-\alpha_{n-1})^i}{i!} \\ &= \left(\sum_{i=1}^{k-1}\frac{\mathrm{d}^i}{\mathrm{d}u^i}q_j(\alpha_{n-1})\frac{1}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l\right)e_n + \frac{\mathrm{d}^k}{\mathrm{d}u^k}q_j(c)\frac{(e_{n-1})^k}{k!} \\ &\leqslant \chi_j(t)e_n(t) + \frac{M}{\gamma^{k-1}}\frac{e_{n-1}^k}{k!} \leqslant \chi_j(t)e_n(t) + \frac{M}{\gamma^{k-1}}\frac{\|e_{n-1}\|^k}{k!}, \end{aligned}$$

where

$$\chi_j(t) = \sum_{i=1}^{k-1}\frac{\mathrm{d}^i}{\mathrm{d}u^i}q_j(\alpha_{n-1})\frac{1}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l, \quad \gamma = \max_{t\in[0,1]}\beta(t) - \min_{t\in[0,1]}\alpha(t).$$

Making use of $(\bar{\mathrm{B}}_3)$, we find that

$$\chi_j(t) \leqslant \sum_{i=1}^{k-1}\frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{i!}\sum_{l=0}^{i-1}e_{n-1}^{i-1-l}a_{n-1}^l \leqslant \sum_{i=1}^{k-1}\frac{M}{(\beta-\alpha)^{i-1}}\frac{1}{(i-1)!}(\beta-\alpha)^{i-1} < 3M < 1.$$

Thus, we can find $\lambda < 1$ such that $\chi_j(t) \leqslant \lambda$, $t \in [0,1]$ and consequently, we have

$$\begin{cases} e_n(0) - \mu_1 e_n'(0) = \int_0^1 [q_1(u(s)) - Q_1^*(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\ \qquad\qquad \leqslant \lambda\int_0^1 e_n(s)\,\mathrm{d}s + \dfrac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k, \\ e_n(1) + \mu_2 e_n'(1) = \int_0^1 [q_2(u(s)) - Q_2^*(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\ \qquad\qquad \leqslant \lambda\int_0^1 e_n(s)\,\mathrm{d}s + \dfrac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k. \end{cases} \tag{3.20}$$

By virtue of (3.19) and (3.20), we have

$$\begin{aligned} e_n(t) &= \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma} + \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\int_0^1 [q_1(u(s)) - Q_1^*(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\ &\quad + \frac{(1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\int_0^1 [q_2(u(s)) - Q_2^*(\alpha_n(s),\alpha_{n-1}(s))]\,\mathrm{d}s \\ &\quad + \int_0^1 G(t,s)[f(s,u(s)) - h(t,\alpha_n,\alpha_{n-1})]\,\mathrm{d}s \\ &\leqslant \frac{-(1-\sigma\mu_2)\mathrm{e}^{-\sigma} + \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\lambda\int_0^1 e_n(s)\,\mathrm{d}s + \frac{M}{\gamma^{k-1}k!}\int_0^1 e_{n-1}^k(s)\,\mathrm{d}s\right] \\ &\quad + \frac{(1+\sigma\mu_1) - \mathrm{e}^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)\mathrm{e}^{-\sigma}}\left[\lambda\int_0^1 e_n(s)\,\mathrm{d}s + \frac{M}{\gamma^{k-1}k!}\int_0^1 e_{n-1}^k(s)\,\mathrm{d}s\right] \\ &\quad - \int_0^1 G(t,s)[e_n''(s) + \sigma e_n'(s)]\,\mathrm{d}s \\ &\leqslant \lambda\int_0^1 e_n(s)\,\mathrm{d}s + \frac{M}{\gamma^{k-1}k!}\int_0^1 e_{n-1}^k(s)\,\mathrm{d}s + \frac{N}{k!}\|e_{n-1}\|^k\int_0^1 G(t,s)\,\mathrm{d}s \\ &\leqslant \lambda\|e_n\| + \frac{M}{\gamma^{k-1}k!}\|e_{n-1}\|^k + \frac{N_1}{k!}\|e_{n-1}\|^k = \lambda\|e_n\| + N_2\|e_{n-1}\|^k, \end{aligned}$$

where $N_2 = (M + \gamma^{k-1} N_1)/\gamma^{k-1} k!$ and N_1 is a bound on $N \int_0^1 G(t, s)$. Taking the maximum over [0, 1] and solving the above expression algebraically, we obtain

$$\|e_n\| \leqslant \frac{N_2}{1-\lambda} \|e_{n-1}\|^k.$$

This completes the proof. □

Example. Consider the boundary value problem

$$u''(t) + \sigma u'(t) - te^{u+1} - 2u = 0, \quad t \in [0, 1], \quad \sigma < 0,$$
$$u(0) - \mu_1 u'(0) = \int_0^1 (cu(s) - 1)/2 \, \mathrm{d}s,$$
$$u(1) + \mu_2 u'(1) = \int_0^1 (cu(s) + 1) \, \mathrm{d}s, \tag{3.21}$$

where $\mu_1 \leqslant (1/2 - c/4)$, $\mu_2 \geqslant c/2$, $0 \leqslant c < 1$. It can easily be verified that $\alpha(t) = -1$ and $\beta(t) = t$ are, respectively, lower and upper solutions of (3.21). Also the assumptions of Corollary 2.1 are satisfied. Hence we can obtain a monotone sequence $\{\alpha_n\}$ of approximate solutions converging uniformly and quadratically (rapidly) to the unique solution of the problem (3.21).

4. Conclusions

We have developed an algorithm for the analytic solution of the forced Duffing equation subject to integral boundary conditions. The results established in this paper provide a diagnostic tool to predict the possible onset of diseases such as cardiac disorder and chaos in speech by varying the nonlinear forcing functions $f(t, u)$ and $q_i(u)$ appropriately in (2.1). The present study is equally useful in other applied sciences as mentioned in the introduction of the paper. If the nonlinearity $f(t, u)$ in the forced Duffing equation is of convex type, then the assumption (A_2) in Theorem 3.1 reduces to $f_{uu}(t, u) \geqslant 0$ and (B_2) in Theorem 3.2 becomes $(\partial^k/\partial u^k) f(t, u) \geqslant 0$ (that is, $\phi(t, u) = 0$ in this case). The existence results for Duffing equation with Dirichlet boundary conditions can be recorded by taking $q_1(\cdot) = 0 = q_2(\cdot)$ and $\mu_1 = 0 = \mu_2$ in (2.1) and in fact this fixation improves the results obtained in [42,15]. Further, for $q_1(\cdot) = a$, $q_2(\cdot) = b$ (a and b are constants) and $\mu_1 = 0 = \mu_2$ in (2.1), our results become the existence results for Duffing equation with nonhomogeneous Dirichlet boundary conditions and thereby generalize the work presented in [5]. If we take $\mu_1 = 0 = \mu_2$ in (2.1), our problem reduces to the Dirichlet boundary value problem involving the forced Duffing equation with integral boundary conditions. In case, we fix $q_1(\cdot) = a$, $q_2(\cdot) = b$ in (2.1), the existence results for the forced Duffing equation with separated boundary conditions appear as a special case of our main results and also generalize the results of [33]. Thus, several interesting observations can be presented by choosing the parameters and functions involved in (2.1) appropriately.

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