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GENERALIZED QUASILINEARIZATION METHOD FOR A FORCED DUFFING EQUATION WITH THREE–POINT NONLINEAR BOUNDARY CONDITIONS

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Abstract. We develop a generalized quasilinearization method for a forced Duffing equation with three-point nonlinear boundary conditions and obtain two monotone sequences of approximate solutions converging quadratically to the unique solution of the problem.

1. Introduction

The method of quasilinearizaion (QSL) provides an adequate approach for obtaining approximate solutions of nonlinear problems. The origin of the quasilinearizaion lies in the theory of dynamic programming [1-3]. This method applies to semilinear equations with convex (concave) nonlinearities and generates a monotone scheme whose iterates converge quadratically to the solution of the problem at hand. The assumption of convexity proved to be a stumbling block for the further development of the method. The nineties brought new dimensions to this technique. The most interesting new idea was introduced by Lakshmikantham [4-5] who generalized the method of quasilinearizaion by relaxing the convexity assumption. This development was so significant that it attracted the attention of many researchers and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations, see [6-17] and references therein. Some real-world applications of the quasilinearization technique can be found in [18-22].

Multi-point nonlinear boundary value problems, which refer to a different family of boundary conditions in the study of disconjugacy theory [23], have been addressed by many authors, for example, see [24-26]. In this paper, we study a generalized quasilinearization method for a forced Duffing equation with nonlinear three-point boundary conditions. In fact, two monotone sequences of upper and lower solutions converging quadratically to the unique solution of the problem are presented.

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2. Preliminaries

We consider a three-point boundary value problem for the forced Duffing equation given by

$$x'' + kx' + f(t,x) = 0, \tag{2.1}$$

$$x(0) = a, \quad x(1) = g(x(1/2)), \tag{2.2}$$

where f is continuous with $f_x < 0$ on $[0,1] \times R$ and $g : R \longrightarrow R$ is continuous.

Here, we remark that the three-point nonlinear boundary conditions 2.2 [15] give rise to Dirichlet boundary conditions for $a = 0$ and $g = 0$ which have been addressed in [27-28] whereas $g =$ constant corresponds to a nonhomogeneous Dirichlet boundary value problem [29].

By Green's function method, the solution, $x(t)$ of (2.1)–(2.2) can be written as

$$x(t) = a\left[\frac{e^{-k} - e^{-kt}}{e^{-k} - 1}\right] + g(x(1/2))\left[\frac{1 - e^{-kt}}{1 - e^{-k}}\right] + \int_0^1 G_k(t,s)f(s,x(s))ds,$$

where

$$G_k(t,s) = \begin{cases} \frac{e^{ks}}{k(1-e^k)}[1 - e^{k(1-s)}][1 - e^{-kt}], & 0 \leqslant t \leqslant s, \\ \frac{e^{ks}}{k(1-e^k)}[1 - e^{k(1-t)}][1 - e^{-ks}], & s \leqslant t \leqslant 1. \end{cases}$$

We say that $\alpha \in C^2[0,1]$ is a lower solution of the boundary value problem (2.1)–(2.2) if

$$\alpha''(t) + k\alpha'(t) + f(t,\alpha(t)) \geqslant 0, \quad \mathrm{t} \in [0,1],$$
$$\alpha(0) \leqslant a, \quad \alpha(1) \leqslant g(\alpha(1/2)),$$

and $\beta \in C^2[0,1]$ is an upper solution of (2.1)–(2.2) if

$$\beta''(t) + k\beta'(t) + f(t,\beta(t)) \leqslant 0, \quad t \in [0,1]$$

$$\beta(0) \geqslant a, \quad \beta(1) \geqslant g(\beta(1/2)).$$

Now, we present comparison and existence results related to (2.1)–(2.2) which play a pivotal role in proving the main result.

THEOREM 2.1. *Assume that f is continuous with $f_x < 0$ on $[0,1] \times R$ and g is continuous on R satisfying a one-sided Lipschitz condition: $g(x) - g(y) \leqslant L(x-y)$, $0 \leqslant L < 1$. Let β and α be the upper and lower solutions of* (2.1)–(2.2) *respectively. Then* $\alpha(t) \leqslant \beta(t)$, $t \in [0,1]$.

Proof. Define $h(t) = \alpha(t) - \beta(t)$. For the sake of contradiction, we suppose that $h(t) > 0$ for some $t \in [0,1]$. First we take $t_0 \in (0,1)$. Then by the definition of lower and upper solutions and the assumption $f_x < 0$, we obtain

$$\begin{aligned} h''(t_0) + kh'(t_0) &= \alpha''(t_0) + k\alpha'(t_0) - \beta''(t_0) - k\beta'(t_0) \\ &\geqslant -f(t_0,\alpha(t_0)) + f(t_0,\beta(t_0)) > 0. \end{aligned}$$

Now, employing a standard procedure [30] in the applications of upper and lower solutions, let $h(t)$ have a local positive maximum at $t_0 \in (0,1)$, then $h'(t_0) = 0$ and

$h''(t_0) \leqslant 0$, which contradicts the above inequality. Thus, for $t_0 \in (0,1)$, we have $\alpha(t) \leqslant \beta(t)$. Now, suppose that $h(t)$ has a local positive maximum at $t_0 = 1$, then $h'(1) = 0$ and $h''(1) < 0$. On the other hand, using the definition of lower and upper solutions together with the fact that g satisfies a one sided Lipschitz condition, we find that

$$h(1) \;=\; \alpha(1) - \beta(1) \leqslant g(\alpha(\frac{1}{2})) - g(\beta(\frac{1}{2})) < \alpha(\frac{1}{2}) - \beta(\frac{1}{2}) = h(\frac{1}{2}),$$

which is a contradiction. Similarly, we get a contradiction for $t_0 = 0$. Hence we conclude that $\alpha(t) \leqslant \beta(t)$ on $[0,1]$. □

THEOREM 2.2. *Assume that f is continuous on $[0,1] \times R$ with $f_x < 0$ and g is continuous on R satisfying a one-sided Lipschitz condition: $g(x) - g(y) \leqslant L(x-y)$, $0 \leqslant L < 1$. Further, we assume that there exist an upper solution β and a lower solution α of* (2.1)–(2.2) *such that $\alpha(t) \leqslant \beta(t)$, $t \in [0,1]$. Then there exists a solution $x(t)$ of* (2.1)–(2.2) *satisfying $\alpha(t) \leqslant x(t) \leqslant \beta(t)$, $t \in [0,1]$.*

Proof. Let us define F and G by

$$F(t,x) = \begin{cases} f(t,\beta) - \dfrac{x-\beta}{1+x-\beta}, & \text{if } x(t) > \beta(t), \\ f(t,x), & \text{if } \alpha(t) \leqslant x(t) \leqslant \beta(t), \\ f(t,\alpha) - \dfrac{x-\alpha}{1+|x-\alpha|}, & \text{if } x(t) < \alpha(t), \end{cases}$$

$$\hat{g}(x) = \begin{cases} g(\beta(\frac{1}{2})), & \text{if } x > \beta(\frac{1}{2}), \\ g(x), & \text{if } \alpha(\frac{1}{2}) \leqslant x \leqslant \beta(\frac{1}{2}), \\ g(\alpha(\frac{1}{2})), & \text{if } x < \alpha(\frac{1}{2}). \end{cases}$$

Since $F(t,x)$ and $\hat{g}(x)$ are continuous and bounded, a standard application of Schauder's fixed point theorem ensures the existence of a solution, x of the problem

$$x''(t) + kx'(t) + F(t,x(t)) = 0, \quad t \in [0,1],$$
$$x(0) = a, \quad x(1) = \hat{g}(x(1/2)).$$

In order to complete the proof, we need to show that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ on $[0,1]$. For that, we set $h(t) = \alpha(t) - x(t)$ and observe that $h(0) \leqslant 0$. For the sake of the contradiction, let $h(t) > 0$ for some $t \in (0,1]$. We define

$$t_0 = \inf\{\tau \in [0,1] : h(\tau) \geqslant h(t),\ 0 \leqslant t \leqslant 1\},$$

and note that $0 < t_0$ by continuity. As $\hat{g}$ satisfies a one-sided Lipschitz condition on $[\alpha(\frac{1}{2}), \beta(\frac{1}{2})]$, it follows that

$$h(1) \;=\; \alpha(1) - x(1) \leqslant \hat{g}(\alpha(1/2)) - \hat{g}(x(1/2)) < (\alpha(1/2) - x(1/2)) = h(1/2).$$

As in the proof of Theorem 2.1, let $h(t)$ have a local maximum at $t_0 \in (0,1)$ implying that $h'(t_0) = 0$ and $h''(t_0) \leqslant 0$. On the other hand, by the definition of upper and lower solutions together with the assumption $F_x < 0$, we have

$$\begin{aligned} h''(t_0) + kh'(t_0) &= \alpha''(t_0) + k\alpha'(t_0) - (x''(t_0) + kx'(t_0)) \\ &\geqslant -F(t_0, \alpha(t_0)) + F(t_0, x(t_0)) > 0. \end{aligned}$$

This contradicts our supposition. Hence $\alpha(t) - x(t) \leqslant 0$. Similarly, it can be shown that $x(t) \leqslant \beta(t)$. Thus, it follows that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$, $t \in [0,1]$. □

3. Main Result

THEOREM 3. *Assume that*

(A_1) α_0, β_0 *are lower and upper solutions of* (2.1)–(2.2) *respectively.*

(A_2) $f(t,x) \in C([0,1] \times R)$ *be such that* $f_x < 0$ *and* $(f_{xx}(t,x) + \phi_{xx}(t,x)) \geqslant 0$, *where* $\phi_{xx}(t,x)) \geqslant 0$ *for some continuous function* $\phi(t,x)$ *on* $[0,1] \times R$.

(A_3) g *is continuous on* R *such that* g', g'' *exist and* $0 \leqslant g' < 1$, $(g''(x) + \psi''(x)) \leqslant 0$ *with* $\psi'' \leqslant 0$ *on* R *for some continuous function* $\psi(x)$.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *that converge quadratically in the space of continuous functions on* $[0,1]$ *to the unique solution* x *of* (2.1)–(2.2).

Proof. Define $F : [0,1] \times R \to R$ by

$$F(t,x) = f(t,x) + \phi(t,x),$$

and $G : R \to R$ by

$$G(x) = g(x) + \psi(x).$$

Using the generalized mean value theorem together with (A_2) and (A_3), we obtain

$$f(t,x) \geqslant f(t,y) + F_x(t,y)(x-y) + \phi(t,y) - \phi(t,x), \tag{3.1}$$

$$g(x) \leqslant g(y) + G'(y)(x-y) + \psi(y) - \psi(x). \tag{3.2}$$

Now, we set

$$F(t,x;\alpha_0) = f(t,\alpha_0) + F_x(t,\alpha_0)(x-\alpha_0) + \phi(t,\alpha_0) - \phi(t,x),$$

$$\overline{F}(t,x;\alpha_0,\beta_0) = f(t,\beta_0) + F_x(t,\alpha_0)(x-\beta_0) + \phi(t,\beta_0) - \phi(t,x),$$

and

$$\begin{aligned} h(x(1/2);\alpha_0,\beta_0) &= g(\alpha_0(1/2)) + G'(\beta_0(1/2))(x(1/2) - \alpha_0(1/2)) \\ &\quad + \psi(\alpha_0(1/2)) - \psi(x(1/2)), \\ \hat{h}(x(1/2);\beta_0) &= g(\beta_0(1/2)) + G'(\beta_0(1/2))(x(1/2) - \beta_0(1/2)) \\ &\quad + \psi(\beta_0(1/2)) - \psi(x(1/2)). \end{aligned}$$

Consider the BVPs

$$x''(t) + kx'(t) + F(t,x;\alpha_0) = 0, \quad t \in [0,1], \tag{3.3}$$

$$x(0) = a, \quad x(1) = h(x(1/2); \alpha_0, \beta_0), \tag{3.4}$$

and

$$x''(t) + kx'(t) + \overline{F}(t, x; \alpha_0, \beta_0) = 0, \quad t \in [0,1], \tag{3.5}$$

$$x(0) = a, \quad x(1) = \hat{h}(x(1/2), \beta_0). \tag{3.6}$$

Let us show that α_0 and β_0 are respectively lower and upper solutions of (3.3)–(3.4). By definition of lower solution and the fact that $F(t, \alpha_0; \alpha_0) = f(t, \alpha_0)$, we get

$$\alpha_0'' + k\alpha_0' + F(t, \alpha_0; \alpha_0) = \alpha_0'' + k\alpha_0' + f(t, \alpha_0) \geqslant 0,$$
$$\alpha_0(0) \leqslant a, \quad \alpha_0(1) \leqslant g(\alpha_0(1/2)) = h(\alpha_0(1/2); \alpha_0; \beta_0),$$

which implies that α_0 is a lower solution of (3.3)–(3.4). Using (3.1) and the definition of upper solution, we have

$$\begin{aligned} &\beta_0'' + k\beta_0' + F(t, \beta_0; \alpha_0) \\ &\quad = \beta_0'' + k\beta_0' + f(t, \alpha_0) + F_x(t, \alpha_0)(\beta_0 - \alpha_0) + \phi(t, \alpha_0) - \phi(t, \beta_0) \\ &\quad \leqslant \beta_0'' + k\beta_0' + f(t, \beta_0) \leqslant 0. \end{aligned}$$

Moreover, $\beta_0(0) \geqslant a$ and there exists $c_0 \in (\alpha_0(1/2), \beta_0(1/2))$ such that

$$\begin{aligned} &g(\beta_0(1/2)) - h(\beta_0(1/2); \alpha_0, \beta_0) \\ &\quad = g(\beta_0(1/2)) - g(\alpha_0(1/2)) - G'(\beta_0(1/2))(\beta_0(1/2) - \alpha_0(1/2)) \\ &\qquad\qquad - \psi(\alpha_0(1/2)) + \psi(\beta_0(1/2)) \\ &\quad = G(\beta_0(1/2)) - G(\alpha_0(1/2)) - G'(\beta_0(1/2))(\beta_0(1/2) - \alpha_0(1/2)) \\ &\quad = [G'(c_0) - G'(\beta_0(1/2))](\beta_0(1/2) - \alpha_0(1/2)) \geqslant 0. \end{aligned}$$

Thus, β_0 is an upper solution of (3.3)–(3.4). Hence, by Theorem 2.2, there is a solution α_1 of (3.3)–(3.4) satisfying

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta_0(t), \quad t \in [0,1]. \tag{3.7}$$

Note that Theorem 2.2 applies since $h' = g'(\beta_0(1/2))$. Similarly, β_0 is an upper solution of (3.5)–(3.6) as

$$\overline{F}(t, \beta_0; \alpha_0; \beta_0) = f(t, \beta_0), \quad g(\beta_0(1/2)) = \hat{h}(\beta_0(1/2); \beta_0).$$

As before, using (3.1), we obtain

$$\begin{aligned} &\alpha_0'' + k\alpha' + \overline{F}(t, \alpha_0; \alpha_0, \beta_0) \\ &\quad = \alpha_0'' + k\alpha' + f(t, \beta_0) + F_x(t, \alpha_0)(\alpha_0 - \beta_0) + \phi(t, \beta_0) - \phi(t, \alpha_0) \\ &\quad \geqslant \alpha_0'' + k\alpha' + f(t, \alpha_0) \geqslant 0. \end{aligned}$$

Also, $\alpha_0(0) \leqslant a$ and there exists $c_1 \in (\alpha_0(1/2), \beta_0(1/2))$ such that

$$\begin{aligned} &\hat{h}(\alpha_0(1/2); \beta_0) - g(\alpha_0(1/2)) \\ &\quad = g(\beta_0(1/2)) - g(\alpha_0(1/2)) + \psi(\beta_0(1/2)) - \psi(\alpha_0(1/2)) \\ &\qquad\qquad + G'(\beta_0(1/2))(\alpha_0(1/2) - \beta_0(1/2)) \\ &\quad = G(\beta_0(1/2)) - G(\alpha_0(1/2)) \\ &\qquad\qquad + G'(\beta_0(1/2))(\alpha_0(1/2) - \beta_0(1/2)) \\ &\quad = [G'(c_1) - G'(\beta_0(1/2))](\beta_0(1/2) - \alpha_0(1/2)) \geqslant 0. \end{aligned}$$

Thus, α_0 is a lower solution of (3.5)–(3.6). Again, by Theorem 2.2, there exists a solution β_1 of (3.5)–(3.6) such that

$$\alpha_0(t) \leqslant \beta_1(t) \leqslant \beta_0(t), \quad t \in [0,1]. \tag{3.8}$$

Now, we show that $\alpha_1 \leqslant \beta_1$. To do this we prove that α_1, β_1 are lower and upper solutions of (2.1)–(2.2) respectively. Using the fact that α_1 is a solution of (3.3)–(3.4), we get

$$\begin{aligned}
&\alpha_1''(t) + k\alpha_1'(t) + f(t,\alpha_1) \\
&\qquad \geqslant \ \alpha_1''(t) + k\alpha_1'(t) + f(t,\alpha_0) + F_x(t,\alpha_0)(\alpha_1 - \alpha_0) + \phi(t,\alpha_0) - \phi(t,\alpha_1) \\
&\qquad = \ \alpha_1''(t) + k\alpha_1'(t) + F(t,\alpha_1;\alpha_0) = 0, \\
&\alpha_1(0) \ = \ a,
\end{aligned}$$

and

$$\begin{aligned}
&g(\alpha_1(1/2)) - \alpha_1(1) \\
&\qquad = g(\alpha_1(1/2)) - g(\alpha_0(1/2)) - G'(\beta_0(1/2))(\alpha_1(1/2) - \alpha_0(1/2)) \\
&\qquad\qquad - \psi(\alpha_0(1/2)) + \psi(\alpha_1(1/2)) \\
&\qquad = [G'(c_2) - G'(\beta_0(1/2))](\alpha_1(1/2) - \alpha_0(1/2)) \geqslant 0,
\end{aligned}$$

where $c_2 \in (\alpha_0(1/2), \alpha_1(1/2))$. This implies that α_1 is a lower solution of (2.1)–(2.2). Similarly, it can be shown that β_1 is an upper solution of (2.1)–(2.2). By Theorem 2.1, it follows that

$$\alpha_1(t) \leqslant \beta_1(t), \quad t \in [0,1]. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) yields

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta_1(t) \leqslant \beta_0(t), \quad t \in [0,1].$$

Continuing this process, by induction, one can prove that

$$\alpha_n(t) \leqslant \alpha_{n+1}(t) \leqslant \beta_{n+1}(t) \leqslant \beta_n(t), \quad t \in [0,1], \quad n = 0,1,...,$$

where α_{n+1} satisfies the problem

$$\begin{gathered}
x''(t) + kx'(t) + F(t,x;\alpha_n) = 0, \quad t \in [0,1]. \\
x(0) = a, \quad x(1) = h(x(1/2);\alpha_n,\beta_n),
\end{gathered}$$

and β_{n+1} satisfies the BVP

$$\begin{gathered}
x''(t) + kx'(t) + \overline{F}(t,x;\alpha_n,\beta_n) = 0, \quad t \in [0,1], \\
x(0) = a, \quad x(1) = h(x(1/2);\beta_n).
\end{gathered}$$

Since $[0,1]$ is compact and the convergence is monotone, it follows that the convergence of each sequence $\{\alpha_n\}$ and $\{\beta_n\}$ is uniform. Employing the standard arguments [15, 20], we conclude that x is the limit point of each of the two sequences and consequently, we get

$$x(t) = a\left[\frac{e^{-k} - e^{-kt}}{e^{-k} - 1}\right] + g(x(1/2))\left[\frac{1 - e^{-kt}}{1 - e^{-k}}\right] + \int_0^1 G_k(t,s)f(s,x(s))ds.$$

This proves that x is the unique solution of (2.1)–(2.2).

In order to prove that each of the sequences $\{\alpha_n\}, \{\beta_n\}$ converges quadratically, we set $q_n = \beta_n - x \geqslant 0, \quad p_n = x - \alpha_n \geqslant 0,$ where x denotes the unique solution of (2.1)–(2.2). We only show the quadratic convergence with p_n as the details for the quadratic convergence for q_n are similar. Applying the mean value theorem, there exist $\alpha_n \leqslant c_3, c_4 \leqslant x$ and $\alpha_n \leqslant \zeta_1 \leqslant \alpha_{n+1}$ such that

$$\begin{aligned}
&p''_{n+1} + kp'_{n+1}\\
&\quad= -f(t,x)+f(t,\alpha_n)+F_x(t,\alpha_n)(\alpha_{n+1}-\alpha_n)+\phi(t,\alpha_n)-\phi(t,\alpha_{n+1})\\
&\quad= -f_x(t,c_3)(x-\alpha_n)+F_x(t,\alpha_n)(\alpha_{n+1}-x+x-\alpha_n)-\phi_x(t,\zeta_1)(\alpha_{n+1}-\alpha_n)\\
&\quad= [-F_x(t,c_3)+F_x(t,\alpha_n)+\phi_x(t,c_3)-\phi_x(t,\zeta_1)]p_n+[-F_x(t,\alpha_n)+\phi_x(t,\zeta_1)]p_{n+1}\\
&\quad\geqslant [-F_x(t,x)+F_x(t,\alpha_n)+\phi_x(t,\alpha_n)-\phi_x(t,x)]p_n+[-F_x(t,\zeta_1)+\phi_x(t,\zeta_1)]p_{n+1}\\
&\quad= -F_{xx}(t,c_4)p_n^2-\phi_{xx}(t,c_3)p_n^2-f_x(t,\zeta_1)p_{n+1}\\
&\quad\geqslant -M\|p_n\|^2,
\end{aligned}$$

where A is a bound on $\|F_{xx}\|$, B is a bound on $\|\phi_{xx}\|$ for $t \in [0,1]$ and $M = A+B$. Here $\|.\|$ denotes the supremum norm on $C[0,1]$. Also there exist $\alpha_n(1/2) \leqslant c_5, c_6 \leqslant x$, $c_5 \leqslant c_7 \leqslant \beta_n(1/2)$ and $\alpha_n \leqslant \zeta_2 \leqslant \alpha_{n+1}$ such that

$$\begin{aligned}
P_{n+1}(t) &= [g(x(1/2)) - h(\alpha_{n+1}(1/2);\alpha_n,\beta_n)]\left(\frac{1 - e^{-kt}}{1 - e^{-k}}\right)\\
&\qquad + \int_0^1 G_k(t,s)[f(s,x) - F(s,\alpha_{n+1};\alpha_n)]ds\\
&= [g(x(1/2)) - g(\alpha_n(1/2) - G'(\beta_n(1/2))(\alpha_{n+1} - \alpha_n(1/2))\\
&\qquad -\psi(\alpha_n(1/2))+\psi(\alpha_{n+1}(1/2))]\left(\frac{1-e^{-kt}}{1-e^{-k}}\right)-\int_0^1 G_k(t,s)[p''_{n+1}+kp'_{n+1}]ds\\
&\leqslant [(G'(c_5) - G'(\beta_n(1/2)) - (\psi'(c_6) - \psi'(\zeta_2)))p_n(1/2)\\
&\qquad + (G'(\beta_n(1/2))-\psi'(\zeta_2))p_{n+1}(1/2)]\left(\frac{1-e^{-kt}}{1-e^{-k}}\right)+M\|p_n\|^2\int_0^1 |G_k(t,s)|ds\\
&\leqslant [-G''(c_7)(\beta_n(1/2)-c_5)p_n(1/2)+g'(\beta_n(1/2))p_{n+1}(1/2)]\left[\frac{1-e^{-kt}}{1-e^{-k}}\right]+M_1\|p_n\|^2\\
&\leqslant [-G''(c_7)(\beta_n(1/2)-\alpha_n(1/2))p_n(1/2)+g'(\beta_n(1/2))p_{n+1}(1/2)]+M_1\|p_n\|^2\\
&= [-G''(c_7)(q_n(1/2)+p_n(1/2))p_n(1/2)+g'(\beta_n(1/2))p_{n+1}(1/2)]+M_1\|p_n\|^2\\
&\leqslant M_2(\frac{1}{2}q_n^2(1/2) + \frac{3}{2}p_n^2(1/2)) + g'(\beta_n(1/2))p_{n+1}(1/2) + M_1\|p_n\|^2\\
&\leqslant (\frac{3}{2}M_2 + M_1)\|p_n\|^2 + \frac{M_2}{2}\|q_n\|^2 + \lambda\|p_{n+1}\|,
\end{aligned}$$

where $\|g'\| \leqslant \lambda < 1, M_1$ provides a bound on $M \int_0^1 |G_k(t,s)|ds, M_2$ provides a bound on $\|G''\|$. Letting $M_3 = \frac{3}{2}M_2 + M_1,\ M_4 = \frac{M_2}{2}$ and solving algebraically for $\|p_{n+1}\|$, we obtain

$$\|p_{n+1}\| \leqslant \frac{1}{1-\lambda}[M_3\|p_n\|^2 + M_4\|q_n\|^2].$$

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REFERENCES

[1] R. BELLMAN, *Methods of Nonlinear Analysis*, Vol. 2, Academic Press, New York, 1973.
[2] R. BELLMAN AND R. KALABA, *Quasilinearization and Nonlinear Boundary Value Problems*, Amer. Elsevier, New York, 1965.
[3] E. S. LEE, *Quasilinearization and Invariant Embedding*, Academic Press, New York, 1968.
[4] V. LAKSHMIKANTHAM, *An extension of the method of quasilinearization*, J. Optim. Theory Appl. 82(1994), 315-321.
[5] V. LAKSHMIKANTHAM, *Further improvement of generalized quasilinearization*, Nonlinear Anal. 27(1996), 223-227.
[6] B. AHMAD, *A quasilinearization method for a class of integro-differential equations with mixed nonlinearities*, Nonlinear Anal.: Real World Appl. (In Press).
[7] B. AHMAD, R. A. KHAN AND S. SIVASUNDARAM, *Generalized quasilinearization method for a first order differential equation with integral boundary condition*, Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal. 12(2005), 289-296.
[8] B. AHMAD, A. AL-SAEDI AND S. SIVASUNDARAM, *Approximation of the solution of nonlinear second order integro-differential equations*, Dynamic Systems Appl. 14 (2005), 253-263.
[9] R. A. KHAN AND R. RODRIGUEZ-LOPEZ, *Existence and approximation of solutions of second order nonlinear four point boundary value problems*, Nonlinear Anal. 63(2005), 1094-1115.
[10] V. LAKSHMIKANTHAM AND A. S. VATSALA, *Generalized quasilinearization versus Newton's method*, Appl. Math. Comput. 164(2005), 523-530.
[11] F. M. ATICI AND S. G. TOPAL, *The generalized quasilinearization method and three-point boundary value problems on time scales*, Appl. Math. Lett. 18(2005), 577-585.
[12] A. BUICA, *Quasilinearization method for nonlinear elliptic boundary value problems*, J. Optim. Theory Appl. 124(2005), 323-338.
[13] A. R. ABD-ELLATEEF KAMAR AND Z. DRICI, *Generalized quasilinearization method for systems of nonlinear differential equations with periodic boundary conditions*, Dynam. Contin. Discrete Impuls. Systems Ser. A Math. Anal. 12(2005), 77-85.
[14] T. JANKOWSKI, *Quadratic approximation of solutions for differential equations with nonlinear boundary conditions*, Compu. Math. Appl. 47(2004), 1619-1626.
[15] P. ELOE AND Y. GAO, *The method of quasilinearization and a three-point boundary value problem*, J. Korean Math. Soc. 39(2002), 319-330.
[16] B. AHMAD, J. J. NIETO AND N. SHAHZAD, *The Bellman-Kalaba-Lakshamikantham quasilinearization method for Neumann problems*, J. Math. Anal. Appl. 257(2001), 356-363.
[17] A. CABADA AND J. J. NIETO, *Quasilinearization and rate of convergence for higher order nonlinear periodic boundary value problems*, J. Optim. Theory Appl. 108(2001), 97-107.
[18] S. NIKOLOV, S. STOYTCHEV, A. TORRES AND J. J. NIETO, *Biomathematical modeling and analysis of blood flow in an intracranial aneurysms*, Neurological Research 25(2003), 497-504.
[19] J. J. NIETO AND A. TORRES, *A nonlinear biomathematical model for the study of intracranial aneurysms*, J. Neurological Science 177(2000), 18-23.
[20] V. LAKSHMIKANTHAM AND A. S. VATSALA, *Generalized Quasilinearization for Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1998.
[21] YERMACHENKO AND F. SADYRBAEV, *Quasilinearization and multiple solutions of the Emden-Fowler type equation*, Math. Model. Anal. 10(2005), 41-50.

[22] V. B. MANDELZWEIG AND F. TABAKIN, *Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs*, Computer Physics Comm. 141(2001), 268-281.
[23] W. COPPEL, *Disconjugacy, Lecture Notes in Mathematics*, Vol. 220, Springer-Verlag, NewYork/Berlin (1971).
[24] I. T. KIGURADZE AND A. G. LOMTATIDZE, *On certain boundary value problems for second-order linear ordinary differential equations with singularities*, J. Math. Anal. Appl. 101(1984), 325-347.
[25] C. P. GUPTA, *A second order m-point boundary value problem at resonance*, Nonlinear Anal. 24(1995), 1483-1489.
[26] C. P. GUPTA AND S. TROFIMCHUCK, *A priori estimates for the existence of a solution for a multi-point boundary value problem*, J. Inequal. Appl. 5(2000), 351-365.
[27] J. J. NIETO, *Generalized quasilinearization method for a second order differential equation with Dirichlet boundary conditions*, Proc. Amer. Math. Soc. 125(1997), 2599-2604.
[28] A. CABADA, J. J. NIETO AND RAFAEL PITA-DA-VEIGE, *A note on rapid convergence of approximate solutions for an ordinary Dirichlet problem*, Dynam. Contin. Discrete Impuls. Systems 4(1998), 23-30.
[29] BASHIR AHMAD AND HUMDA HUDA, *Generalized quasilinearization method for nonlinear nonhomogeneous Dirichlet boundary value problems*, Int. J. Math. Game Theory Algebra 12(2002), 461-467.
[30] L. JACKSON, *Boundary value problems for ordinary differential equations (Studies in Ordinary Differential Equations, Ed. J. K. Hale)*, MAA Studies in Mathematics, Mathematical Association of America, 14(1977), 93-127.

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