Existence of Solutions of the Forced Duffing Equation with Non-convex Integral Boundary Conditions

Ahmed Alsaedi
Department of Mathematics,
Faculty of Science, King Abdulaziz University,
P.O. Box. 80257, Jeddah 21589, Saudi Arabia
E-mail: *aalsaedi@hotmail.com*

Abstract

In this paper, we apply the generalized quasilinearization technique to obtain a monotone sequence of approximate solutions converging monotonically and quadratically to the unique solution of the forced Duffing equation with non-convex type integral boundary conditions.

Key words: Duffing equation, integral boundary conditions, quasilinearization, quadratic convergence, higher order convergence.
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1 Introduction

Boundary value problems involving integral boundary conditions have received considerable attention, see for instance, [1-8] and references therein. Integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics, see for example, [9-11].
In this paper, we apply the generalized quasilinearization technique to obtain a monotone sequence of approximate solutions that converges quadratically to the unique solution of the forced Duffing equation with non-convex type integral boundary conditions. The importance of the work lies in the fact that the convexity/concavity assumption on the nonlinear functions in integral boundary conditions has been relaxed. The method of quasilinearization provides an elegant and easier approach for obtaining sequences of approximate solutions converging monotonically and quadratically to the unique solution of the problem at hand. For the details of this method, see [12-21] and the references therein.

2 Preliminaries and basic results

Consider the following boundary value problem

$$\begin{cases} u''(t)+\sigma u'(t)+f(t,u)=0, \quad 0<t<1,\ \sigma\in\mathbb{R}-\{0\}, \\ u(0)-\mu_1 u'(0)=\int_0^1 h_1(u(s))ds, \quad u(1)+\mu_2 u'(1)=\int_0^1 h_2(u(s))ds, \end{cases} \tag{2.1}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $h_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are continuous functions and μ_i are nonnegative constants. Clearly the homogenous problem

$$u''(t)+\sigma u'(t)=0,\quad 0<t<1,$$
$$u(0)-\mu_1 u'(0)=0,\quad u(1)+\mu_2 u'(1)=0,$$

has only the trivial solution. Therefore, by Green's function method, the solution of (2.1) can be written as

$$u(t)=G_1(t)+\int_0^1 G(t,s)f(s,u(s))ds,$$

where

$$\begin{aligned} G_1(t) &= \frac{1}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}} \\ &\times \Big[((-1+\sigma\mu_2)e^{-\sigma}+e^{-\sigma t})\int_0^1 h_1(u(s))ds+((1+\sigma\mu_1)-e^{-\sigma t})\int_0^1 h_2(u(s))ds\Big], \end{aligned}$$

and

$$G(t,s)=\Lambda\begin{cases} [(1-\sigma\mu_2)-e^{\sigma(1-s)}][(1+\sigma\mu_1)-e^{-\sigma t}], & 0\le t\le s,\\ [(1-\sigma\mu_2)-e^{\sigma(1-t)}][(1+\sigma\mu_1)-e^{-\sigma s}], & s\le t\le 1,\end{cases}$$

$$\Lambda=\frac{e^{\sigma s}}{\sigma[(1-\sigma\mu_2)-(1+\sigma\mu_1)e^{\sigma}]}.$$

We note that $G(t,s)>0$ on $(0,1)\times(0,1)$.

Definition 2.1. A function $\alpha\in C^2[0,1]$ is a lower solution of (2.1) if

$$\alpha''(t)+\sigma\alpha'(t)+f(t,\alpha(t))\ge 0,\quad 0<t<1,$$
$$\alpha(0)-\mu_1\alpha'(0)\le\int_0^1 h_1(\alpha(s))ds,\quad \alpha(1)+\mu_2\alpha'(1)\le\int_0^1 h_2(\alpha(s))ds.$$

Similarly, $\beta\in C^2[0,1]$ is an upper solution of (2.1) if the inequalities in the definition of lower solution are reversed.
We need the following known results [21] to prove the main result.

Theorem 2.1. Let α and β be lower and upper solutions of the boundary value problem (2.1) respectively. Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be such that $f_u(t,u)<0$ and $h_i:\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying one sided Lipschitz condition: $h_i(u)-h_i(v)\le L_i(u-v),\ 0\le L_i<1,\ i=1,2$. Then $\alpha(t)\le\beta(t)$.

Theorem 2.2. Assume that α and β are lower and upper solutions of the boundary value problem (2.1) respectively such that $\alpha(t)\le\beta(t)$. If $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $h_i:\mathbb{R}\to\mathbb{R}$ are continuous and satisfy one sided Lipschitz condition, then there exists a solution $u(t)$ of (2.1) such that $\alpha(t)\le u(t)\le\beta(t),\ t\in[0,1]$.

3 Main result

Theorem 3.1. Assume that

($\mathbf{A_1}$) α and $\beta \in C^2[0,1]$ are respectively lower and upper solutions of (2.1) such that $\alpha(t) \leq \beta(t), \quad t \in [0,1]$.

($\mathbf{A_2}$) $f(t,x) \in C^2([0,1] \times \mathbb{R})$ be such that $f_x < 0$ and $(f_{xx}(t,x) + \phi_{xx}(t,x)) \geq 0$, where $\phi_{xx}(t,x)) \geq 0$ for some continuous function $\phi(t,x)$ on $[0,1] \times \mathbb{R}$.

($\mathbf{A_3}$) $h_i \in C^2(\mathbb{R})$ $(i = 1,2)$ are nondecreasing, $0 \leq h_i'(x) < 1$ and $h_i''(x) + \psi_i''(x) \geq 0$, for some continuous function $\psi(x)$ on $\mathbb{R}$.

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to the unique solution of the problem.

Proof. Define $F : [0,1] \times \mathbb{R} \to \mathbb{R}, \quad \mu : \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \phi(t,x)$ and $\mu_i(x) = h_i(x) + \psi_i(x)$. Using the generalized mean value theorem together with (A_2), we obtain

$$f(t,x) \geq f(t,y) + F_x(t,y)(x-y) + \phi(t,y) - \phi(t,x), \tag{3.1}$$

$$h_i(x) \geq h_i(y) + \mu_i'(y)(x-y) + \psi_i(y) - \psi_i(x), \tag{3.2}$$

where $x, y \in \mathbb{R}$. Now, we set

$$g(t,x,y) = f(t,y) + F_x(t,y)(x-y) + \phi(t,y) - \phi(t,x), \tag{3.3}$$

$$H_i(x,y) = h_i(y) + \mu_i'(y)(x-y) + \psi_i(y) - \psi_i(x), \tag{3.4}$$

and note that $g_x(t,x,y) < 0$, $0 \leq \frac{\partial}{\partial x} H_i(x,y) < 1$, and

$$\begin{cases} f(t,x) \geq g(t,x,y), \\ f(t,x) = g(t,x,x), \end{cases} \tag{3.5}$$

$$\begin{cases} h_i(x) \geq H_i(x,y), \\ h_i(x) = H_i(x,x). \end{cases} \tag{3.6}$$

Fixing $\alpha = w_0$, we consider the problem

$$\begin{aligned} x''(t) + kx'(t) + g(t,x,w_0) &= 0, \quad t \in [0,1] \\ x(0) - k_1 x'(0) &= \int_0^1 H_1(x(s), w_0(s))ds, \\ x(1) + k_2 x'(1) &= \int_0^1 H_2(x(s), w_0(s))ds. \end{aligned} \tag{3.7}$$

Using (A_1), (3.5) and (3.6), we obtain

$$w_0''(t) + kw_0'(t) + g(t,w_0,w_0) = w_0''(t) + kw_0'(t) + f(t,w_0) \geq 0, \quad t \in [0,1],$$

$$w_0(0) - k_1 w_0'(0) \leq \int_0^1 h_1(w_0(s))ds = \int_0^1 H_1(w_0(s), w_0(s))ds,$$

$$w_0(1) + k_2 w_1'(0) \leq \int_0^1 h_2(w_0(s))ds = \int_0^1 H_2(w_0(s), w_0(s))ds,$$

and

$$\beta''(t) + k\beta'(t) + g(t, \beta, w_0) \leq \beta''(t) + k\beta'(t) + f(t, \beta) \leq 0, \quad t \in [0,1],$$

$$\beta(0) - k_1\beta'(0) \geq \int_0^1 h_1(\beta(s))ds \geq \int_0^1 H_1(\beta(s), w_0(s))ds,$$

$$\beta(1) + k_2\beta'(1) \geq \int_0^1 h_2(\beta(s))ds \geq \int_0^1 H_2(\beta(s), w_0(s))ds,$$

which imply that w_0 and β are respectively lower and upper solutions of (3.7). It follows by Theorems 2.1 and 2.2 that there exists the unique solution w_1 of (3.7) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0,1].$$

Next, consider the problem

$$\begin{aligned} x''(t) + kx'(t) + g(t, x, w_1) &= 0, \quad t \in [0,1], \\ x(0) - k_1 x'(0) &= \int_0^1 H_1(x(s), w_1(s))ds, \\ x(1) + k_2 x'(1) &= \int_0^1 H_2(x(s), w_1(s))ds. \end{aligned} \tag{3.8}$$

Following the earlier procedure, it is straightforward to show that w_1 and β are lower and upper solutions of (3.8) and hence, by Theorems 2.1 and 2.2, there exists the unique solution w_2 of (3.8) such that

$$w_1(t) \leq w_2(t) \leq \beta(t), \quad t \in [0,1].$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq ... \leq w_n \leq \beta(t), \quad t \in [0,1],$$

where the element w_n of the sequence $\{w_n\}$ is a solution of the problem

$$x''(t) + kx'(t) + g(t, x, w_{n-1}) = 0, \quad t \in [0,1],$$

$$x(0) - k_1 x'(0) = \int_0^1 H_1(x(s), w_{n-1}(s))ds,$$

$$x(1) + k_2 x'(1) = \int_0^1 H_2(x(s), w_{n-1}(s))ds,$$

and is given by

$$w_n(t) = P_n(t) + \int_0^1 G(t,s)g(s, w_n(s), w_{n-1}(s))ds, \tag{3.9}$$

where

$$\begin{aligned} P_n(t) &= \frac{-(1-kk_2)e^{-k} + e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 H_1(w_n(s), w_{n-1}(s))ds \\ &+ \frac{(1+kk_1) - e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 H_2(w_n(s), w_{n-1}(s))ds, \end{aligned}$$

Using the fact that $[0,1]$ is compact and the monotone convergence of the sequence $\{w_n\}$ is pointwise, it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence, passing onto the limit $n \to \infty$, (3.9) yields

$$x(t) = P(t) + \int_0^1 G(t,s)f(s, x(s))ds,$$

where

$$\begin{aligned} P(t) &= \frac{-(1-kk_2)e^{-k} + e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 h_1(x(s))ds \\ &+ \frac{(1+kk_1) - e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 h_2(x(s))ds. \end{aligned}$$

Thus, $x(t)$ is a solution of (2.1). Now, we show that the convergence of the sequence is quadratic. For that, we set $e_n(t) = x(t) - w_n(t) \geq 0,\ t \in [0,1]$. Using Taylor's theorem and (3.4), we obtain

$$\begin{aligned} e_n(0) - k_1 e_n'(0) &= \int_0^1 [h_1(x(s)) - H_1(w_n(s), w_{n-1}(s))]ds \\ &= \int_0^1 [h_1(x(s)) - h_1(w_{n-1}(s)) - \mu_1'(w_{n-1}(s))(w_n - w_{n-1}) - \psi_1(w_{n-1}) + \psi_1(w_n)]ds \\ &= \int_0^1 [h_1'(\gamma_1)(x - w_{n-1}) - \mu_1'(w_{n-1})(x - w_{n-1}) + \mu_1'(w_{n-1})(x - w_n) \\ &+ \psi_1'(\gamma_2)(w_n - w_{n-1})]ds \\ &= \int_0^1 \{[h_1'(\gamma_1)e_{n-1} - \mu_1'(w_{n-1})e_{n-1} + \mu_1'(w_{n-1})e_n + \psi_1'(\gamma_2)e_{n-1} - \psi_1'(\gamma_2)e_n]ds \\ &= \int_0^1 \{[h_1'(\gamma_1) - \mu_1'(w_{n-1}) + \psi_1'(\gamma_2)]e_{n-1} + [\mu_1'(w_{n-1}) - \psi_1'(\gamma_2)]e_n\}ds \\ &\leq \int_0^1 \{[\mu_1'(\gamma_1) - \psi_1'(\gamma_1) - \mu_1'(w_{n-1}) + \psi_1'(\gamma_2)]e_{n-1} + [\mu_1'(w_{n-1}) - \psi_1'(w_{n-1})]e_n\}ds \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \{[\mu_1''(\gamma_3)(\gamma_1 - w_{n-1}) - \psi_1'(w_{n-1}) + \psi_1'(\gamma_2)]e_{n-1} + h_1'(w_{n-1})e_n\}ds \\
&\leq \int_0^1 \{[\mu_1''(\gamma_3)(x - w_{n-1}) + \psi_1''(\gamma_4)(x - w_{n-1})]e_{n-1} + h_1'(w_{n-1})e_n\}ds \\
&= \int_0^1 \{[\mu_1''(\gamma_3) + \psi_1''(\gamma_4)]e_{n-1}^2 + h_1'(w_{n-1})e_n\}ds,
\end{aligned}
$$

and

$$
\begin{aligned}
e_n(1) + k_2 e_n'(1) &= \int_0^1 [h_2(x(s)) - H_2(w_n(s), w_{n-1}(s))]ds \\
&= \int_0^1 \{[\mu_2''(\gamma_3^*) + \psi_2''(\gamma_4^*)]e_{n-1}^2 + h_2'(w_{n-1})e_n\}ds,
\end{aligned}
$$

where $w_{n-1} \leq \gamma_3 \leq \gamma_1, \gamma_3^* \leq x$, $w_{n-1} \leq \gamma_4 \leq \gamma_2, \gamma_4^* \leq w_n$. In view of (A_3), there exist $\lambda_i < 1$ and $C_i \geq 0$ such that $h_i'(w_{n-1}(s)) \leq \lambda_i$ and $\mu_i''(\gamma_3(\gamma_3^*)) + \psi_i''(\gamma_4(\gamma_4^*)) \leq C_i \ \ (i = 1, 2)$. Let $\lambda(< 1) = \max\{\lambda_1, \lambda_2\}$ and $C(\geq 0) = \max\{C_1, C_2\}$, then

$$
e_n(0) - k_1 e_n'(0) \leq \lambda \int_0^1 e_n(s)ds + C \int_0^1 e_{n-1}^2(s)ds,
$$

$$
e_n(1) + k_2 e_n'(1) \leq \lambda \int_0^1 e_n(s)ds + C \int_0^1 e_{n-1}^2(s)ds.
$$

In view (A_2) and (3.3), we obtain

$$
\begin{aligned}
& e_n''(t) + k e_n'(t) = x'' + kx' - (w_n + kw_n) \\
=\ & -f(t, x) + g(t, w_n, w_{n-1}) \\
=\ & -f(t, x) + f(t, w_{n-1}) + F_x(t, w_{n-1})(w_n - w_{n-1}) + \phi(t, w_{n-1}) - \phi(t, w_n) \\
=\ & -f_x(t, c_1)(x - w_{n-1}) - F_x(t, w_{n-1})(x - w_n) + F_x(t, w_{n-1})(x - w_{n-1}) \\
-\ & \phi_x(t, c_2)(w_n - w_{n-1}) \\
=\ & [-f_x(t, c_1) + F_x(t, w_{n-1}) - \phi_x(t, c_2)]e_{n-1} + [-F_x(t, w_{n-1}) + \phi_x(t, c_2)]e_n \\
=\ & [-F_x(t, c_1) + F_x(t, w_{n-1}) + \phi_x(t, c_1) - \phi_x(t, c_2)]e_{n-1} \\
+\ & [-F_x(t, w_{n-1}) + \phi_x(t, c_2)]e_n \\
\geq\ & [-F_x(t, x) + F_x(t, w_{n-1}) + \phi_x(t, w_{n-1}) - \phi_x(t, w_n)]e_{n-1} \\
+\ & [-F_x(t, w_{n-1}) + \phi_x(t, w_{n-1})]e_n \\
=\ & [-F_{xx}(t, c_3) - \phi_{xx}(t, c_4)]e_{n-1}^2 - f_x(t, w_{n-1})e_n \\
\geq\ & -[A + B]e_{n-1}^2 \\
=\ & -M\|e_{n-1}\|^2,
\end{aligned}
$$

where $w_{n-1} \leq c_3 \leq c_1 \leq x$, $w_{n-1} \leq c_4 \leq c_2 \leq w_n$, A is a bound on $\|F_{xx}\|$, B is a bound on $\|\phi_{xx}\|$ for $t \in [0,1]$ and $M = A + B$. Thus,

$$\begin{aligned}
e_n(t) = & \frac{-(1-kk_2)e^{-k} + e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 [h_1(x(s)) - H_1(w_n(s), w_{n-1}(s))]ds \\
+ & \frac{(1+kk_1) - e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} \int_0^1 [h_2(x(s)) - H_2(w_n(s), w_{n-1}(s))]ds \\
+ & \int_0^1 G(t,s)[f(s,x(s)) - g(t, w_n, w_{n-1})]ds \\
\leq & \frac{-(1-kk_2)e^{-k} + e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} [\lambda \int_0^1 e_n(s)ds + C \int_0^1 e_{n-1}^2(s)ds] \\
+ & \frac{(1+kk_1) - e^{-kt}}{(1+kk_1) - (1-kk_2)e^{-k}} [\lambda \int_0^1 e_n(s)ds + C \int_0^1 e_{n-1}^2(s)ds] \\
- & \int_0^1 G(t,s)[e_n''(s) + ke_n'(s)]ds \\
\leq & \lambda \int_0^1 e_n(s)ds + C \int_0^1 e_{n-1}^2(s)ds + M\|e_{n-1}\|^2 \int_0^1 G(t,s)ds \\
\leq & \lambda\|e_n\| + C\|e_{n-1}\|^2 + M_1\|e_{n-1}\|^2 = \lambda\|e_n\| + C_1\|e_{n-1}\|^2,
\end{aligned}$$

where $M_1 = Ml$, l is a bound on $\int_0^1 G(t,s)$ and $C_1 = M_1 + C$. Taking the maximum over $[0,1]$, we get

$$\|e_n\| \leq \frac{C_1}{1-\lambda}\|e_{n-1}\|^2.$$

This completes the proof.

4 Concluding remarks

If we take $q_1(.) = a,\ q_2(.) = b$ (a and b are constants) in (2.1), our results correspond to the forced Duffing equation with separated boundary conditions. By taking $\mu_1 = 0 = \mu_2$ in (2.1), our problem reduces to the Dirichlet boundary value problem involving the forced Duffing equation with integral boundary conditions. By taking $\psi(x) \equiv 0$ in the assumption (A_2), the results of [21] appear as a special case of our main result. Thus, the present study is quite useful and improve some earlier results.

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